Ese let $K = \mathbb{C}$.

$V = V_{\mathbb{P}^2}(X_1^2 + X_2^2 - X_0^2) \longrightarrow \mathbb{P}^1$

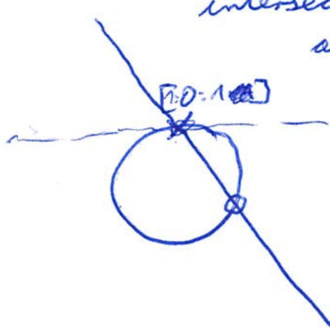
$[x_0 : x_1 : x_2] \longmapsto [x_1 : x_2 - x_0]$ unless $x_1 = x_2 - x_0 = 0$

$[x_2 + x_0 : -x_1]$ unless $x_2 + x_0 = x_1 = 0$

("The slope of the line connecting $[1:0:1]$ and $[x_0 : x_1 : x_2]$")

$[y_0^2 + y_1^2 : -2y_0y_1 : y_0^2 - y_1^2] \longleftarrow [y_0 : y_1]$

("The second pt. of intersection of V and the line of slope $\frac{y_1}{y_0}$ through $[1:0:1]$.")



Rmk Up to a change of basis, the inverse map is the degree 2 Veronese map $\mathbb{P}^1 \longrightarrow \mathbb{P}^2$.

$[2(x_2 - x_0)x_1 : (x_2 - x_0)^2 - x_1^2 : -((x_2 - x_0)^2 + x_1^2)]$

$x_1 y_1 = (x_2 - x_0) y_0$

$x_1^2 + x_2^2 - x_0^2 = 0$

$x_1^2 y_1^2 + x_2^2 y_1^2 - x_0^2 y_1^2 = 0$

$(x_2 - x_0)^2 y_0^2 + (x_2^2 - x_0^2) y_1^2 = 0$

$(x_2 - x_0) y_0^2 = -(x_0 + x_2) y_1^2$

$x_2 + x_0 = 2y_0^2$

$x_2 - x_0 = 2y_1^2$

$(x_0 - x_2) y_0^2 = (x_2 + x_0) y_1^2$

$x_0 = y_0^2 + y_1^2$

$x_0 = y_0^2 - y_1^2$

$x_2 = y_0^2 + y_1^2$

$x_1 = 2y_0 y_1$

15. Images

Rmk We've seen before, that morphisms $\varphi : V \to W$ between alg. subsets V, W of <u>affine</u> space often have a nonclosed image.

(E.g. proj. of $V(XY-1)$ onto X-axis)

This can't happen if V is an alg. subset of <u>projective</u> space:

Thm 15.1 If V is an alg. subset of $\mathbb{P}^n$, then the image of any morphism $\varphi : V \to W$ ($W \subseteq \mathbb{P}^m$) is closed (= algebraic).

Rmk This implies that any such morphism is a closed map.

Pf The graph
$$\Gamma := \{(P, \varphi(P)) \mid P \in V\} = \{(P, Q) \mid Q = \varphi(P)\}$$
is an algebraic subset of $V \times W \subseteq \mathbb{P}^n \times \mathbb{P}^m$. (!)

It is therefore described by polynomials $f_1, \dots, f_k$ in $X_0, \dots, X_n, Y_0, \dots, Y_m$ which are homogeneous in $X_0, \dots, X_n$ and homogeneous in $Y_0, \dots, Y_m$.

Now:

$[b_0 : \dots : b_m] \notin \psi(V)$

$$\Updownarrow$$

$f_1(X_0,\dots,X_n,b_0,\dots,b_m), \dots \in K[X_0,\dots,X_n]$ have no common root in $\mathbb{P}^n$

$$\Updownarrow \text{ Proj. Nsts.}$$

$$\exists\, e \geq 0: \underbrace{(X_0,\dots,X_n)^e \subseteq (f_1(X_0,\dots,b_0,\dots), f_2(\dots), \dots)}$$

$$\Updownarrow$$

Every hom. pol. $g \in K[X_0,\dots,X_n]$ of degree e is a lin. comb.

$$g = \sum f_i(X_0,\dots,b_0,\dots)\, h_i(X_0,\dots) \qquad (I)$$

with $h_i \in K[X_0,\dots,X_n]$.

(If f_i is hom. of degree d_i in $X_0,\dots,X_n$, taking the hom. degree e part of (I), w.l.o.g. h_i is hom. of degree $e-d_i$.)

Let F_d be the vector space of hom. deg. d pol. in $X_0,\dots,X_n$.

$$\Updownarrow$$

The linear map

$$F_{e-d_1} \times \dots \times F_{e-d_k} \longrightarrow F_e$$

$$(h_1,\dots,h_k) \longmapsto \sum_i f_i(X_0,\dots,b_0,\dots)\, h_i(X_0,\dots)$$

is surjective.

$$\Updownarrow$$

Let $q = \dim(F_e)$.

Some $q \times q$-minor of the matrix representing this linear map is $\neq 0$.

Note: The entries of the matrix are (hom.) pol. in $b_0,\dots,b_m$ (of the same degree), so the set of such $(b_0,\dots,b_m)$ is an open subset of K^{m+1}. □

Cor 15.2 If $V \subseteq \mathbb{P}^n$ is an irreducible alg. subset, all morphisms $\varphi : V \to K^m$ are constant (im. $= \{pt\}$).

Pf W.l.o.g. $m = 1$. We have a morphism $\varphi : V \to \cancel{K}\,\mathbb{P}^1$, whose image doesn't contain ∞.

$\varphi(V) \subseteq \cancel{K}\,\mathbb{P}^1$ is closed.

$\Rightarrow \varphi(V) = \cancel{K}\,\mathbb{P}^1$ or $\varphi(V)$ is finite

(impossible)

$\Downarrow$ V irred. $\Rightarrow \overline{\varphi(V)}$ irred.

$\varphi(V)$ is a point.

$\square$

__Cor 15.3__ ~~Thm~~ If V is an alg. subset of $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$, then any morphism $\varphi: V \to W$ is closed.

__Pf (sketch)__ Compose with the Segre map

$$\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k} \longrightarrow \mathbb{P}^{(n_1+1)\cdots(n_k+1)-1},$$

which is an isomorphism onto its image. (Apply the Thm to the image of V in the RHS.) $\square$

Cor 15.4 The join J of any disjoint alg. subsets V, W of $\mathbb{P}^n$ is closed (= algebraic).

~~J is the image of~~

$\cancel{V \times W \times \mathbb{P}^1 \longrightarrow \mathbb{P}^n}$

$\cancel{(P, Q, [u:t]) \longmapsto}$

Warning

$$V \times W \times \mathbb{P}^1 \longrightarrow \mathbb{P}^n$$

$$([P],[Q],[\lambda:\mu]) \longmapsto [\lambda P + \mu Q]$$

is not a well-defined map. (Think about what happens if you rescale P!)

Pf

J is the image of the alg. set

$$\{(P,Q,R) \in V \times W \times \mathbb{P}^n \mid P, Q, R \text{ colinear}\}$$

under the proj. $(P,Q,R) \longmapsto R$. $\square$