14 ~~B~~.6. Dimension

Def The dimension of an (irred.) alg. set $\emptyset \neq V \subseteq \mathbb{P}^n_K$ is the largest length d of a chain
$$V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_d \subseteq V$$
of irred. alg. subsets V_i.

Thm 14 ~~B~~.6.1

For any alg. $\emptyset \neq V \subseteq \mathbb{P}^n_K$:

a) $\dim(V) = \dim(C(V)) - 1$

b) If V is irreducible and $\varphi: K^n \longrightarrow \mathbb{P}^n_K$ is an affine patch with $\varphi^{-1}(V) \neq \emptyset$
$$\dim(V) = \dim(\varphi^{-1}(V)).$$

Rmk b) can fail if V is reducible:
e.g. $V = \{\text{point}\} \cup (\text{line at infinity})$
$\Rightarrow \varphi^{-1}(V) = \{\text{point}\}$.

Pf (for V irred.) ~~By Thm 3.5.2, we can assume that V is irreducible (even in a))~~.

For any chain

$$V_0 \subsetneq \cdots \subsetneq V_d \subseteq V \text{ of irred. sets,}$$

we obtain a chain

$$\{0\} = \ell(\emptyset) \subsetneq \ell(V_0) \subsetneq \cdots \subsetneq \ell(V_d) \subseteq \ell(V) \text{ of irred. sets.}$$

$$\Rightarrow \dim(\ell(V)) \geq \dim(V) + 1$$

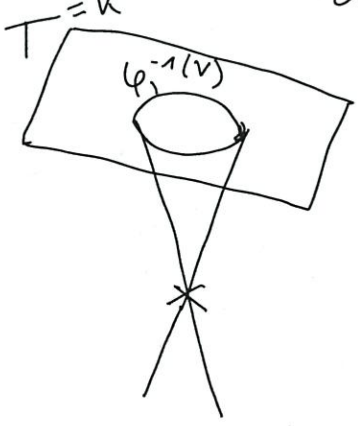
For any chain

$$W_0 \subsetneq \cdots \subsetneq W_d \subseteq \varphi^{-1}(V) \text{ of irred. sets,}$$

we obtain a chain

$$\overline{\varphi(W_0)} \subsetneq \cdots \subsetneq \overline{\varphi(W_d)} \subseteq V \text{ of irred. sets.}$$

$$\Rightarrow \dim(V) \geq \dim(\varphi^{-1}(V)).$$



Let $0 \notin T \subseteq K^{n+1}$ be an n-dim. affine lin. subspace corr. to φ. Then $\ell(V)$ is the Zariski closure of the join of $\{0\}$ and $\ell(V) \cap T \cong \varphi^{-1}(V)$.

By problem 4 on pset 8, we then have $\dim(\ell(V)) = \dim(\varphi^{-1}(V)) + 1$. $\square$

Def Let $V, W \subseteq \mathbb{P}^n$ be irreducible alg. sets.

$$\begin{aligned}\operatorname{codim}(V, W) &:= \dim(W) - \dim(V)\\ &= \dim(\mathcal{C}(W)) - \dim(\mathcal{C}(V))\\ &= \operatorname{codim}(\mathcal{C}(V), \mathcal{C}(W)).\end{aligned}$$

Rmk $\operatorname{codim}(V, W)$ is the largest $d \geq 0$ s.t. there is a chain $V = V_0 \subsetneq \dots \subsetneq V_d = W$ of irred. alg. sets.

Thm 14.6.2 Let $V_1, V_2 \subseteq \mathbb{P}^n$ be irreducible alg. sets with $\operatorname{codim}(V_1, \mathbb{P}^n) + \operatorname{codim}(V_2, \mathbb{P}^n) \leq n$.

Then, a) $V_1 \cap V_2 \neq \emptyset$ and

b) every irred. comp. A of $V_1 \cap V_2$ satisfies

$$\operatorname{codim}(A, \mathbb{P}^n) \leq \operatorname{codim}(V_1, \mathbb{P}^n) + \operatorname{codim}(V_2, \mathbb{P}^n).$$

Pf Apply Cor 13.4.3 to the affine cones:

Any irred. comp. B of $\mathcal{C}(V_1) \cap \mathcal{C}(V_2) = \mathcal{C}(V_1 \cap V_2)$

satisfies $\operatorname{codim}(B, K^{n+1}) \leq \operatorname{codim}(\mathcal{C}(V_1), K^{n+1}) + \operatorname{codim}(\mathcal{C}(V_2), K^{n+1})$

$= \operatorname{codim}(V_1, \mathbb{P}^n) + \operatorname{codim}(V_2, \mathbb{P}^n)$

$\leq n.$

This shows b) since $\mathcal{C}(A)$ is an irred. comp. of $\mathcal{C}(V_1 \cap V_2)$.

For a), note that $0 \in \mathcal{C}(V_1) \cap \mathcal{C}(V_2)$, so $\mathcal{C}(V_1) \cap \mathcal{C}(V_2) \neq \emptyset$, so $\mathcal{C}(V_1 \cap V_2)$ contains at least one irred. comp. B, which, as we have seen above, has dimension $\dim(B) \geq 1$. $\Rightarrow B \neq \{0\}$, so $\mathcal{C}(V_1 \cap V_2) \neq \emptyset$. $\square$

Ex Any two curves in $\mathbb{P}^2$ intersect.

Any curve and surface in $\mathbb{P}^3$ intersect.

Any three surfaces in $\mathbb{P}^3$ intersect.

14.7. Products

For details, see Chapters 4 and 5 in Shafarevich (Basic Algebraic Geometry 1).

Rmk $K^n \times K^m = K^{n+m}$, but $\mathbb{P}^n \times \mathbb{P}^m$ is not $\mathbb{P}^{n+m}$!

Def A set $V \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is algebraic if for each pair of (standard) chart maps φ, ψ of $\mathbb{P}^n, \mathbb{P}^m$, the subset

$\{(P, Q) \mid (\varphi(P), \psi(Q)) \in V\}$ of $K^n \times K^m = K^{n+m}$

is algebraic.

Exe $V := \{([x_0 : x_1 : x_2], [y_0 : y_1]) : x_0^3 y_0 y_1 + 2 x_1 x_2^2 y_1^2 = 0\}$

is an algebraic subset of $\mathbb{P}^2 \times \mathbb{P}^1$.

More generally:

Rmk $V \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is algebraic if and only if there are polynomials $f_1, \dots, f_k \in K[x_0, \dots, x_n, y_0, \dots, y_m]$ which are homogeneous in $x_0, \dots, x_n$ and homogeneous in $y_0, \dots, y_m$ (not necessarily of the same degree) such that

$V = \{([x_0 : \dots : x_n], [y_0 : \dots : y_m]) : \forall i : f_i(x_0, \dots, x_n, y_0, \dots, y_m) = 0\}$.

(Note: If f is hom. of deg. d in $x_0, \dots, x_n$ and of deg. e in $y_0, \dots, y_m$, then

$f(\lambda x_0, \dots, \lambda x_n, \mu y_0, \dots, \mu y_m) = \lambda^d \mu^e f(x_0, \dots, x_n, y_0, \dots, y_m)$,

so whether $f(x_0, \dots, x_n, y_0, \dots, y_m) = 0$ only depends on the points $[x_0 : \dots : x_n] \in \mathbb{P}^n$ and $[y_0 : \dots : y_m] \in \mathbb{P}^m$, not on the choice of projective coordinates.)

__Def__ Let $V \subseteq \mathbb{P}^n \times \mathbb{P}^m$, $W \subseteq \mathbb{P}^N \times \mathbb{P}^M$ be alg. subsets.

A __morphism__ $\alpha : V \to W$ is a map such that for all (standard) chart maps $\varphi_1, \varphi_2, \psi_1, \psi_2$ on $\mathbb{P}^n, \mathbb{P}^m, \mathbb{P}^N, \mathbb{P}^M$, the map

$$K^n \times K^m \supseteq \varphi^{-1}(V) \dashrightarrow K^N \times K^M$$

$$(P, Q) \longmapsto \psi^{-1}(\alpha(\varphi(P, Q)))$$

is a rational map, defined wherever the above map makes sense, i.e. wherever $\alpha(\varphi(P,Q)) \in \psi(K^N \times K^M)$.

Here, we let $\varphi(P,Q) = (\varphi_1(P,Q), \varphi_2(P,Q))$
and $\psi(P,Q) = (\psi_1(P,Q), \psi_2(P,Q))$.

$$\begin{tikzcd}
\varphi^{-1}(V) \arrow[r, dashed] \arrow[d, "\varphi"'] & K^N \times K^M \arrow[d, "\psi"] \\
V \arrow[r, "\alpha"'] & W
\end{tikzcd}$$

We similarly define alg. subsets of the product of any number of projective spaces and morphisms between any two such subsets.

Ese

$$\alpha : V^{\subseteq \mathbb{P}^n} \longrightarrow \mathbb{P}^m$$

$$[x_0 : \dots : x_n] \longmapsto [f_0(x_0,\dots,x_n) : \dots : f_m(x_0,\dots,x_n)]$$

is a (well-defined) morphism if $f_0,\dots,f_m \in K[X_0,\dots,X_n]$

are homogeneous polynomials of the same degree d which have no common root on V:

$\mathcal{V}_V(f_0,\dots,f_m) = \emptyset$:

Using the 0-th std. chart maps on $\mathbb{P}^n$ and $\mathbb{P}^m$, the corr. rational map is given by

$$(x_1,\dots,x_n) \longmapsto \left(\frac{f_1(1,x_1,\dots,x_n)}{f_0(1,x_1,\dots,x_n)}, \dots, \frac{f_m(1,x_1,\dots,x_n)}{f_0(1,x_1,\dots,x_n)} \right)$$

(when $f_0(1,x_1,\dots,x_n) \neq 0$).

<u>Warning</u> The "projection" $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, $[x_0 : x_1 : x_2] \mapsto [x_0 : x_1]$ is not well-defined

at $[0:0:1]$.

<u>Warning</u> Not every morphism is of the above form!

You might need different hom. pol. at different points, just like a rat. fct. $f = \frac{a}{b}$ might be def. at points with $a(P) = b(P) = 0$ and there might not be a single choice of a, b that works for all points where f is defined. (HW).

For example:

<u>Def</u> Let $n \geq 1$. The <u>Veronese map</u> of degree $d \geq 1$ is the map $\mathbb{P}^n \longrightarrow \mathbb{P}^{m-1}$ defined by the $m := \binom{n+d-1}{d}$ monomials in $x_0, \dots, x_n$ of degree d.

<u>Ex</u>

$$\begin{aligned} \mathbb{P}^1 &\longrightarrow \mathbb{P}^3 \\ [x_0 : x_1] &\longmapsto [x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3] \end{aligned}$$

is the degree 3 Veronese map on $\mathbb{P}^1$.

Another example:

<u>Def</u> Let $n, m \geq 1$. The <u>Segre map</u> is the map

$$\begin{aligned} \mathbb{P}^n \times \mathbb{P}^m &\longrightarrow \mathbb{P}^{(n+1)(m+1)-1} \\ ([x_0 : \dots : x_n], [y_0 : \dots : y_m]) &\longmapsto [x_0 y_0 : x_0 y_1 : \dots : x_n y_m] \end{aligned}$$

def. by the products $x_i y_j$.