Warning Let $I = (f_1, \dots, f_m)$.

Then, S is the set of homogenizations of elements of I. Unfortunately, the homogenizations $\tilde{f}_1, \dots, \tilde{f}_m$ don't always suffice!

Ex $I = (x_1^2 + x_2,\ x_1) \quad = (x_2, x_1)$

$$\begin{array}{ccc}
x_1^2 + x_0 x_2 = 0,\ x_1 = 0 & & x_2 = 0,\ x_1 = 0 \\
\Updownarrow & & \\
x_0 x_2 = 0,\ x_1 = 0 & & \text{one point} \\
 & & [1:0:0] \\
\text{two points} & & \\
{[0:0:1]},\ [1:0:0] & &
\end{array}$$

Thm 14.2.6 Let $f \in K[x_1, \dots, x_n]$ with homogenization $\tilde{f}$ at x_0. Then, $\overline{\varphi_0(V(f))} = V_{\mathbb{P}^n_K}(\tilde{f})$.

Pf If $g = fh \in (f)$, then $\tilde{g} = \tilde{f}\tilde{h}$. $\Rightarrow \overline{\varphi_0(V(f))} = V_{\mathbb{P}^n}(\{\tilde{f}\tilde{h} \mid h \in K[x_1, \dots, x_n]\}) = V_{\mathbb{P}^n}(\tilde{f})$. $\square$

~~Pf "$\subseteq$" clear~~

~~"$\supseteq$" Let $g \in (f)$ with homogenization $\tilde{g} = \tilde{f}\tilde{h}$~~

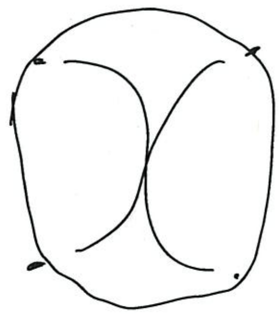
~~$g = fh,\ h \in K[x_1, \dots, x_n]$~~ ~~Lemma 3.2.4~~

~~If $\tilde{f}(P) = 0$, then $\tilde{g}(P) = 0$.~~

~~$\Rightarrow V_{\mathbb{P}^n_K}(\tilde{f}) = V_{\mathbb{P}^n_K}(\{\text{hom. } \tilde{g} \text{ of } g \in (f)\}) = \overline{\varphi_0(V(f))}$.~~ ~~$\square$~~

Cor 14.2.7 Any affine chart $\varphi : K^n \hookrightarrow \mathbb{P}^n_K$ is an open map (sending open sets to open sets).

Pf



Let $U = K^n \setminus A$ be open in K^n.

$\Rightarrow \varphi(U) = \mathbb{P}^n_K \setminus \left(\left(\mathbb{P}^n_K \setminus \mathrm{im}(\varphi) \right) \cup \overline{\varphi(U)} \right)$

is open in $\mathbb{P}^n_K$.

$\square$

Cor 14.2.8 A subset $A \subseteq \mathbb{P}^n_K$ is alg. if and only if $\varphi_i^{-1}(A) \subseteq K^n$ is alg. for all standard affine charts φ_i.

$\rightsquigarrow$ You obtain the topology on $\mathbb{P}^n_K$ by glueing together the topologies on the affine charts.

14

~~B~~.3. Vanishing ideals

Def An ideal $I \subseteq K[x_0, \dots, x_n]$ is homogeneous if it is generated by (finitely many) homogeneous polynomials.

Thm ~~B~~.3.1 (14) I is hom. if and only if for every $d \geq 0$ and $f \in I$, the degree d part f_d also lies in I.

Pf "$\Leftarrow$" $f = \sum_d f_d$

$\Rightarrow$ I is gen. by the hom. parts of the elements of I

"$\Rightarrow$" Let $I = (g_1, \dots, g_m)$ with g_i hom. of degree d_i.

Let $f \in I$ with degree d part f_d.

Write $f = \sum_i g_i h_i$ with $h_1, \dots, h_m \in K[x_0, \dots, x_n]$.

Let $h_{i,e}$ be the degree e part of h_i.

$$\Rightarrow f_d = \sum_i \underbrace{g_i}_{\text{hom. of deg. } d_i} \underbrace{h_{i,d-d_i}}_{\text{deg. } d-d_i} \in I.$$

$\square$

Def For any homogeneous ideal $I \subseteq K[X_0,\dots,X_n]$, we let $\mathcal{V}_{\mathbb{P}^n_K}(I) := \mathcal{V}_{\mathbb{P}^n_K}(\{f \in I \text{ homogeneous}\})$.

Rmk $\mathcal{V}_{\mathbb{P}^n_K}(\text{ideal gen. by } S) = \mathcal{V}_{\mathbb{P}^n_K}(S)$ for any set S of hom. pol.

Rmk $C(\mathcal{V}_{\mathbb{P}^n_K}(I)) = \{0\} \cup \mathcal{V}_{K^{n+1}}(I)$

Def The vanishing ideal of a subset $A \subseteq \mathbb{P}^n_K$ is the ideal $\mathcal{J}_{\mathbb{P}^n}(A) \subseteq K[X_0,\dots,X_n]$ generated by the homogeneous pol. f vanishing on A (s.t. $A \subseteq \mathcal{V}_{\mathbb{P}^n_K}(f)$).

Lemma 14.3.2

If $A \neq \emptyset$, then $\mathcal{J}(A) = \mathcal{J}(C(A))$.

If $A = \emptyset$, then $\mathcal{J}(A) = K[X_0,\dots,X_n]$.

(although $\mathcal{J}(C(A)) = \mathcal{J}(\{0\}) = (X_0,\dots,X_n)$).

Pf $\underline{A = \emptyset}$: clear

$\underline{A \neq \emptyset}$: "$\subseteq$" If a hom. pol. f vanishes on A, it vanishes on $C(A)$.

"$\supseteq$" If a pol. $f \in K[X_0, \dots, X_n]$ vanishes on $C(A) \subseteq K^{n+1}$, so do its homogeneous parts. They must then vanish on A. $\square$

14.4. Projective Nullstellensatz

From now on, we again assume that K is algebraically closed.

Thm 14.4.1 (Weak proj. Nsts)

Let $I \subseteq K[X_0, \dots, X_n]$ be a hom. ideal. Then, the following are equivalent:

a) $V_{\mathbb{P}^n_K}(I) = \emptyset$

b) $(X_0, \dots, X_n) \subseteq \sqrt{I}$

vanishes only at 0 in K^{n+1} ($\Rightarrow$ at no point in $\mathbb{P}^n_K$)

c) $X_0^m, \dots, X_n^m \in I$ for some $m \geq 0$.

Pf b) $\iff$ c): clear

a) $\iff$ b):

$$\mathbb{V}_{\mathbb{P}^n_k}(I)=\emptyset$$

$$\iff C(\mathbb{V}_{\mathbb{P}^n_k}(I)) = \{0\}$$

$$C(\mathbb{V}_{\mathbb{P}^n_k}(I)) = \{0\}\cup \mathbb{V}_{k^{n+1}}(I)$$

$$\iff \mathbb{V}_{k^{n+1}}(I) \subseteq \{0\}$$

$$\iff J(\mathbb{V}_{k^{n+1}}(I)) \supseteq J(\{0\}) = (x_0,\dots,x_n)$$

$$J(\mathbb{V}_{k^{n+1}}(I)) \overset{\text{Hilbert's Nsts}}{=} \sqrt{I}$$

$\square$

Cor 14.4.2 (Proj. Nsts) For any hom. id. I,

$$J(\mathbb{V}_{\mathbb{P}^n_k}(I)) = \begin{cases} \sqrt{I}, & (x_0,\dots,x_n)\not\subseteq\sqrt{I}, \\ K[x_0,\dots,x_n], & (x_0,\dots,x_n)\subseteq\sqrt{I}. \end{cases}$$

Pf second case: $\mathbb{V}_{\mathbb{P}^n_k}(I)=\emptyset \Rightarrow J(\mathbb{V}_{\mathbb{P}^n_k}(I)) = K[x_0,\dots,x_n]$

first case: $J(\mathbb{V}_{\mathbb{P}^n_k}(I)) \underset{\text{Lemma 14.3.2}}{=} J(C(\mathbb{V}_{\mathbb{P}^n_k}(I))) = J(\mathbb{V}_{k^{n+1}}(I)) \underset{\text{Hilbert's Nsts}}{=} \sqrt{I}.$ $\square$

14.5. Irreducibility

<u>Def</u> An alg. subset $\emptyset \neq A \subseteq \mathbb{P}^n_k$ is <u>irreducible</u> if you can't write $A = A_1 \cup A_2$ with any alg. sets $A_1, A_2 \subsetneq A_2$.

<u>Ex</u> One point, $\mathbb{P}^n_K$

<u>Thm 14.5.1</u> Let $A \neq \emptyset$ be an alg. subset of $\mathbb{P}^n_k$.

The following are equivalent:

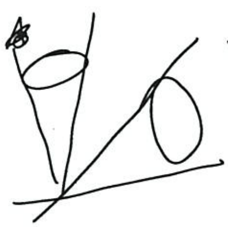
a) A is irreducible.

b) $C(A)$ is irreducible.

c) $I^J(A)$ is a prime ideal.

<u>Pf</u> <u>b) $\Leftrightarrow$ c)</u> $I^J(C(A)) = I^J(A)$

<u>b) $\Rightarrow$ a)</u> $A = A_1 \cup A_2$, $A_1, A_2 \subsetneq A$



$C(A) = C(A_1) \cup C(A_2)$, $C(A_1), C(A_2) \subsetneq C(A)$

<u>a) $\Rightarrow$ c)</u> Say $f, g \notin I^J(A)$ with $fg \in I^J(A)$.

Let $\deg(f) = d$ and f_d be the degree d part of f.

Let $\deg(g) = e$ and g_e be the degree e part of g.

W.l.o.g. $f_d, g_e \notin I(A)$.

(Otherwise, replace f by $f - f_d$ or
g by $g - g_e$,
reducing the degree of f or g.)

$\Rightarrow$ $\deg(fg) = d+e$ and $f_d g_e$ is the degree $d+e$ part of fg.

$I(A)$ hom. ideal $\underset{\text{Thm 3.3.1}}{\Rightarrow}$ $f_d g_e \in I(A)$

Take $A_1 = A \cap V_{\mathbb{P}^n_k}(f_d)$,
$A_2 = A \cap V_{\mathbb{P}^n_k}(g_e)$.

$f_d g_e \in I(A) \Rightarrow A_1 \cup A_2 = A$

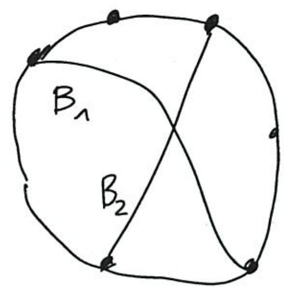
$f_d \notin I(A) \Rightarrow A_1 \subsetneq A$

$g_e \notin I(A) \Rightarrow A_2 \subsetneq A$. $\square$

Thm 14.5.2 Let $A \subseteq \mathbb{P}^n_k$ be irred. and let φ be an affine chart. Then,

$$\varphi^{-1}(A) = \emptyset \quad \text{or } \varphi^{-1}(A) \text{ is irreducible.}$$

Pf If $\emptyset \neq \varphi^{-1}(A) = B_1 \cup B_2$, $B_1, B_2 \subsetneq \varphi^{-1}(A)$,



then

$$A = \overline{\varphi(B_1)} \cup \overline{\varphi(B_2)} \cup \underbrace{(A \setminus \mathrm{im}(\varphi))}_{\text{closed}}$$

with

$$\overline{\varphi(B_1)} \subsetneq A, \ \overline{\varphi(B_2)} \subsetneq A,$$

$$A \setminus \mathrm{im}(\varphi) \subsetneq A. \qquad \square$$

Rmk If $A \neq \emptyset$ and for <u>every</u> affine chart φ, $\varphi^{-1}(A) = \emptyset$ or $\varphi^{-1}(A)$ is irred., then A is irred.

Warning It doesn't suffice to consider just the standard affine charts φ_i. For example $\{[0:1], [1:0]\} \subseteq \mathbb{P}^1_k$ is reducible although the intersections with $U_0 = \{[x_0 : x_1] \mid x_0 \neq 0\}$ and $U_1 = \{[x_0 : x_1] \mid x_1 \neq 0\}$ each consist of just one point.