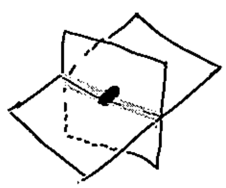
Lemma 14.1.2

Let L be an a-dim. lin. subspace of $\mathbb{P}^n$ and let M be a b-dim. lin. subspace of $\mathbb{P}^n$.
Then, $L \cap M$ is a c-dimensional lin. subspace of $\mathbb{P}^n$ with $c \geq a+b-n$. $\Big(\underbrace{\operatorname{codim}(L\cap M, \mathbb{P}^n)}_{= n-c} \leq \underbrace{\operatorname{codim}(L, \mathbb{P}^n)}_{= n-a} + \underbrace{\operatorname{codim}(M, \mathbb{P}^n)}_{= n-b}\Big)$

Ex If L, M are lines in $\mathbb{P}^2$, then $L \cap M$ is a point or $L = M$.



Pf of Lemma Let $L, M \subseteq \mathbb{P}^n_K$ corr. to $V, W \subseteq K^{n+1}$

$\dim(V) = a+1, \quad \dim(W) = b+1$

$\operatorname{codim}(V, K^{n+1}) = n-a, \ \operatorname{codim}(W, K^{n+1}) = n-b$

$\Rightarrow$ $V \cap W$ is a vector space with
$\operatorname{codim}(V \cap W, K^{n+1}) \leq (n-a) + (n-b)$

$$\Rightarrow \dim(L \cap M) = n - \operatorname{codim}(V\cap W, K^{n+1}) \geq n - (n-a) - (n-b) = a+b-n.$$

$\square$

14.2. Algebraic sets

<u>Def</u> A polynomial $f \in K[X_0, \dots, X_n]$ is <u>homogeneous of degree $d \geq 0$</u> (or a <u>form of degree d</u>) if every monomial in f has degree (exactly) d.

<u>Ex</u> $2X + 3Y$ hom. of deg. 1

<u>Ex</u> $2X + 3Y + 1$ not hom.

<u>Ex</u> $X^3 + 2X^2Y + Y^3$ hom. of degree 3

<u>Ex</u> 0 is homogeneous of <u>every</u> degree $d \geq 0$.

<u>Rmk</u> The hom. degree d pol. form a K-vector space.

<u>Rmk</u> Any pol. $f \in K[X_0, \dots, X_n]$ can be written uniquely as $f = \sum_{d=0}^{m} f_d$ with f_d hom. of degree d (called the <u>degree d part</u> of f).

<u>Rmk</u> If f is hom. of degree d and g is hom. of degree e, then fg is hom. of degree $d+e$.

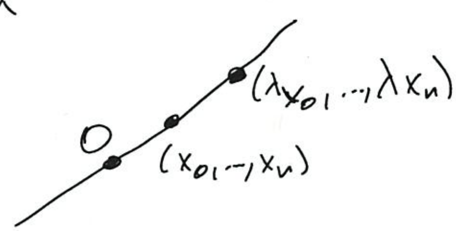
Rmk If $f \in K[Y_0, \dots, Y_m]$ is hom. of degree d and $g_1, \dots, g_m \in K[X_0, \dots, X_n]$ are hom. of degree e, then $f(g_1, \dots, g_m)$ is hom. of degree $d \cdot e$.

Rmk 14.2.1 If f is hom. of degree d, then

$$f(\lambda X_0, \dots, \lambda X_n) = \lambda^d f(X_0, \dots, X_n).$$

Def If $f \in K[X_0, \dots, X_n]$ is hom., we denote by

$$V_{\mathbb{P}^n_K}(f) = \{[x_0 : \dots : x_n] \in \mathbb{P}^n_K \mid f(x_0, \dots, x_n) = 0\}$$



independent of the choice of hom. coord. $x_0, \dots, x_n$ by Rmk 3.2.1!

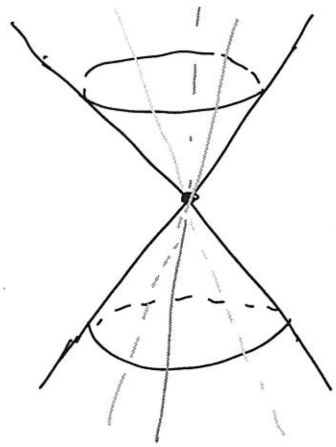
the corresponding set of zeros (the <u>vanishing locus</u> of f).

If $S \subseteq K[X_0, \dots, X_n]$ is a set of hom. pol., let

$$V_{\mathbb{P}^n_K}(S) = \{[x_0 : \dots : x_n] \in \mathbb{P}^n_K \mid f(x_0, \dots, x_n) = 0 \ \forall f \in S\} = \bigcap_{f \in S} V_{\mathbb{P}^n_K}(f).$$

A subset $A = V_{\mathbb{P}^n_K}(S)$ of this form is called <u>algebraic</u>.

Ex $f = x_1^2 + x_2^2 - x_0^2$



$\mathcal{V}(f) = \mathcal{V}_{k^3}(f) =$ cone

$\mathcal{V}_{\mathbb{P}^2}(f) =$ set of lines through 0 on the cone

Rmk $\mathcal{V}_{\mathbb{P}^n_k}(S) \subseteq \mathbb{P}^n_k$ is the set of lines through 0 contained in $\mathcal{V}(S) = \mathcal{V}_{k^{n+1}}(S)$.

Ex Any linear subspace of $\mathbb{P}^n_k$ is algebraic.

Def Let $A \subseteq \mathbb{P}^n_k$ be any subset.

The set $\mathcal{C}(A) = \{0\} \cup \{0 \neq (x_0, \dots, x_n) \in k^{n+1} \mid [x_0 : \dots : x_n] \in A\}$
$\subseteq k^{n+1}$

(the union of $\{0\}$ and the lines in k^{n+1} representing the points in $A \subseteq \mathbb{P}^n_k$)

is called the <u>affine cone</u> of A.

Rmk 14 ~~Lemma 3.2.2~~ If $A \subseteq \mathbb{P}^n_k$ is algebraic, then $\ell(A) \subseteq k^{n+1}$ is algebraic. (def. by the same eqs.)

~~Pf If $A = V_{\mathbb{P}^n_k}(S) \neq \emptyset$, then $\ell(A) = V_{k^{n+1}}(S)$.~~

~~If $A = \emptyset$, then $\ell(A) = \{0\}$.~~ ~~$\square$~~

Rmk $\ell(A \cap B) = \ell(A) \cap \ell(B)$

As before (Lemma 2.2):

Rmk a) $\bigcap_\alpha V_{\mathbb{P}^n}(S_\alpha) = V_{\mathbb{P}^n}\left(\bigcup_\alpha S_\alpha\right)$

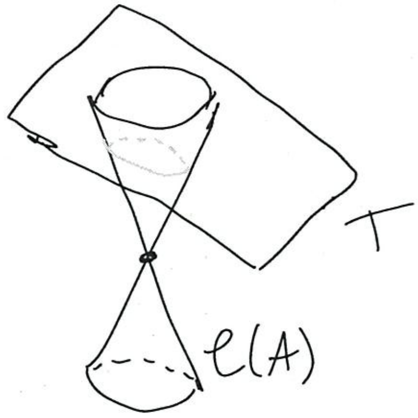
b) $V_{\mathbb{P}^n}(S) \cup V_{\mathbb{P}^n}(T) = V_{\mathbb{P}^n}(\{fg \mid f \in S,\ g \in T\})$

c) $V_{\mathbb{P}^n}(\emptyset) = V_{\mathbb{P}^n}(0) = \mathbb{P}^n$

d) $V_{\mathbb{P}^n}(1) = \emptyset$.

Hence, we again obtain a Zariski topology whose closed sets are the algebraic sets.

Ex Any affine patch $U \subseteq \mathbb{P}^n_K$ (complement of hyperplane) is open.



<u>Lemma 14.2.3</u> Any affine chart $\varphi : K^n \longrightarrow U \subseteq \mathbb{P}^n_K$

is continuous:

If $A \subseteq \mathbb{P}^n_K$ is algebraic, then $\varphi^{-1}(A) \subseteq K^n$ is algebraic.

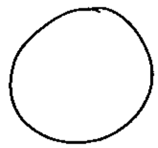
<u>Pf</u> $\varphi^{-1}(A) \subseteq K^n$ "is" the intersection of the affine cone $\mathcal{C}(A)$ with the affine linear subspace T corresponding to the chart. $\square$

<u>Concretely</u> If $\varphi = \varphi_i$ is the i-th standard affine chart, $A = V_{\mathbb{P}^n}(S)$, then

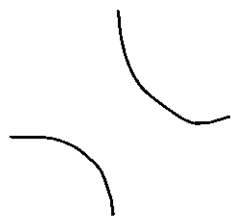
$$\varphi^{-1}(A) = \{(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in K^n \mid f(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) = 0 \ \forall f \in S\}.$$

<u>Ex</u> $A = V_{\mathbb{P}^2}(x_1^2 + x_2^2 - x_0^2)$.

$$\varphi_0^{-1}(A) = V_{K^2}(x_1^2 + x_2^2 - 1)$$



$$\varphi_1^{-1}(A) = V_{K^2}(1 + x_2^2 - x_0^2)$$



The preimage $\varphi^{-1}(A)$ is a conic section for any affine chart φ.

We constructed a map

$$\{\text{alg. subset } A \text{ of } \mathbb{P}^n\} \longrightarrow \{\text{alg. subset } B \text{ of } K^n\}$$
$$A \longmapsto \varphi^{-1}(A)$$

<u>Q</u> How to produce $A \subseteq \mathbb{P}^n_K$ from $B \subseteq K^n$?

<u>A</u> Take $A = \overline{\varphi(B)}$. What are equations defining A?

Def Let $f \in K[X_1, \dots, X_n]$ be a polynomial of degree d and let f_e be its (hom.) degree e. The **homogenization** of $f = \sum_e f_e$ (at X_0) is the hom. degree d pol.

$$\tilde{f} = \sum_e f_e \cdot X_0^{d-e} = X_0^d f\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right)$$

Ex $f = X_1^2 + X_2^2 - 1 \rightsquigarrow \tilde{f} = X_1^2 + X_2^2 - X_0^2$

Note $\tilde{f}(1, X_1, \dots, X_n) = f(X_1, \dots, X_n)$

Lemma 14.2.4 Let $B = V_{K^n}(I)$ for an ideal $I \subseteq K[X_1, \dots, X_n]$. Let $\varphi = \varphi_0$ be the 0-th standard chart of $\mathbb{P}^n_K$. Then,

$\overline{\varphi(B)} = V_{\mathbb{P}^n_K}(S)$ where $S \subseteq K[X_0, \dots, X_n]$ is the set of homogenizations $\tilde{f}$ of the elements $f \in I$ at X_0.

Pf HW.

Cor 14.2.5

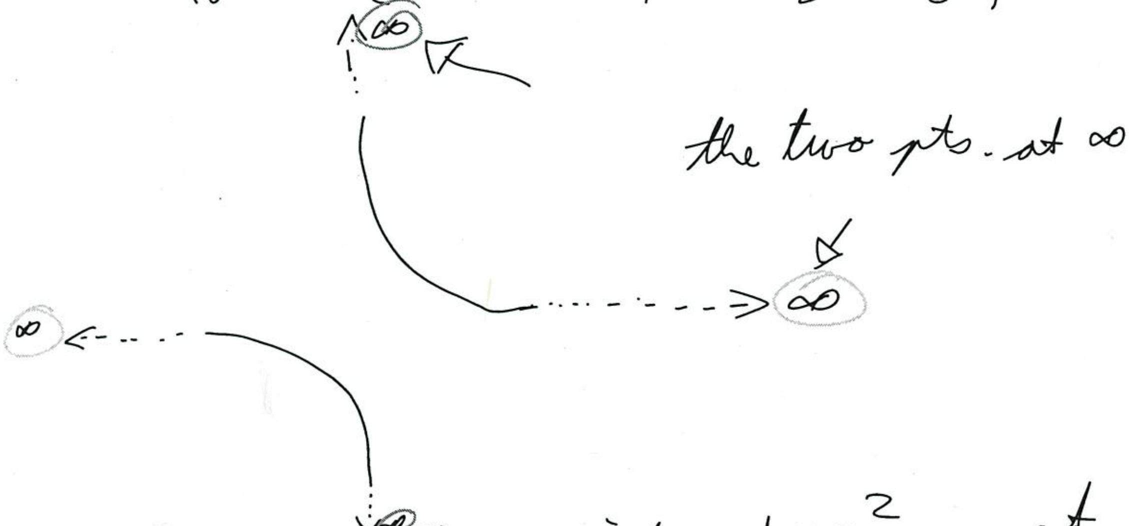
$$\varphi^{-1}\left(\overline{\varphi(B)}\right) = B \text{ for any affine chart.}$$

(We only add points at ∞ to B to obtain $\overline{\varphi(B)}$.)

Pf This follows from the lemma and the previous note. $\square$

Ex $B = \{(x_1, x_2) \mid x_1 x_2 = 1\}$

$$\rightsquigarrow A = \overline{\varphi_0(B)} = \{[x_0 : x_1 : x_2] \mid x_1 x_2 = x_0^2\}$$



What are the points at ∞?

pt. at ∞

$$A \setminus \varphi_0(B) = \{[x_0 : x_1 : x_2] \mid x_1 x_2 = x_0^2,\ x_0 = 0\}$$

$$= \{[0 : x_1 : x_2] \mid x_1 x_2 = 0\}$$

$$= \{[0:0:1], [0:1:0]\}$$