Pf of Cor 13.4.3

Consider the morphism

$$\varphi : V_1 \cap V_2 \longrightarrow V_1 \times V_2 \subseteq K^n \times K^n,$$
$$P \longmapsto (P, P)$$

$$\varphi(V_1 \cap V_2) = \{ (Q,R) \in V_1 \times V_2 \mid Q = R \}$$
$$= \mathcal{V}_{V_1 \times V_2}(X_1 - Y_1, \ldots, X_n - Y_n) =: B$$

if we denote the coord. in $K^n \times K^n$ by $X_1, \ldots, X_n, Y_1, \ldots, Y_n$.

$\varphi : V_1 \cap V_2 \to B$ is an isomorphism with inverse

$$\begin{array}{ccc} K^n \times K^n & & \\ \cup\!\!\!\mid & & \\ V_1 \times V_2 & & K^n \\ \cup\!\!\!\mid & & \cup\!\!\!\mid \\ B & \longrightarrow & V_1 \cap V_2 \\ (Q,R) & \longmapsto & Q \end{array}$$

$\Rightarrow \varphi(A)$ is an irred. comp. of $B = \mathcal{V}_{V_1 \times V_2}(X_1 - Y_1, \ldots, X_n - Y_n)$.

$$\Rightarrow \dim(A) = \dim(\varphi(A)) \overset{\text{Cor 13.2.6}}{\geq} \dim(V_1 \times V_2) - n$$
$$= \dim(V_1) + \dim(V_2) - n$$

$\Rightarrow \operatorname{codim}(A, K^n) \leq \operatorname{codim}(V_1, K^n) + \operatorname{codim}(V_2, K^n).$ $\square$

13.5. Applications, part 2

We obtain a "converse" to Thm 13.3.3:

Thm 13.5.1 For any $n, d \geq 1$ and $m \geq \binom{d+n}{n}$, there are points $P_1, \dots, P_m \in K^n$ s.t. there is no pol. $0 \neq f \in K[x_1, \dots, x_n]$ of degree $\leq d$ with $P_1, \dots, P_m \in \mathcal{V}(f)$.

Pf Let F_d' be the set of pol. of deg. $\leq d$ where at least one coeff. is 1.

Consider the following alg. subset A of $\underbrace{K^n \times \dots \times K^n}_{m} \times F_d'$:

$$A = \{(P_1, \dots, P_m, f) \mid f(P_1) = \dots = f(P_m) = 0\}.$$

We have projections

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "\sigma"] & F_d' \\ K^n \times \dots \times K^n \end{tikzcd}$$

The goal is to show that σ is not surjective, which we'll do by proving $\dim(A) < \dim(K^n \times \dots \times K^n)$.

Recall that $\dim(F_d') = \binom{d+n}{n} - 1$.

Moreover, the preimage of any $f \in F_d'$ in A is

$$\pi^{-1}(f) = \underbrace{\mathcal{V}(f) \times \dots \times \mathcal{V}(f)}_{m} \times \{f\}.$$

Since $f \in F_d'$, we have $f \neq 0$, so $\dim(\mathcal{V}(f)) = n-1$.

$\Rightarrow \dim(\pi^{-1}(f)) = m(n-1).$

Applying Thm 13.4.1 (to the irred. comp. W of F_d' and the irred. comp. V of $\pi^{-1}(W)$), it follows that

$$\dim(A) \leq \binom{d+n}{n} - 1 + m(n-1).$$

Using the assumption that $m \geq \binom{d+n}{n}$, we ~~get~~ indeed get

$$\dim(A) \leq mn - 1 < mn = \dim(K^n \times \dots \times K^n). \qquad \square$$

<u>Remark</u> Applying Prop 13.4.4, we see that in fact

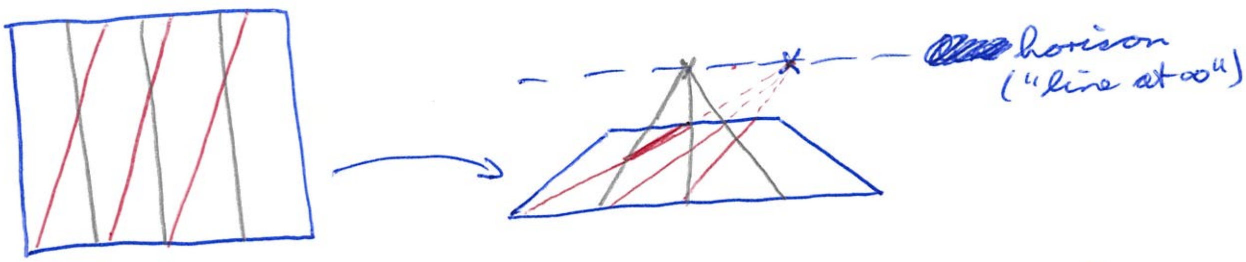
$$\dim(A) = \binom{d+n}{n} - 1 + m(n-1).$$

14. Projective varieties

14.1. Projective space

<u>Motivation</u> Two lines $L_1 \neq L_2$ in K^2 intersect in exactly one point <u>except if they are parallel</u>.

<u>Idea</u> Pretend they intersect in a point at ∞. (infinitely far away)



Any line goes through exactly one additional point at ∞ which depends on the direction (slope) of the line.

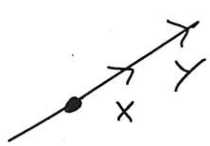
In this section, K can be any field (not necc. alg. closed).

<u>Def</u> The n-dimensional <u>projective space</u> $\mathbb{P}^n_K$ over K is the set of lines in K^{n+1} through the origin. We call the elements of $\mathbb{P}^n_K$ the <u>points</u> in $\mathbb{P}^n_K$.

We denote the line spanned by $\cancel{(0,\dots,0)\neq}\ (x_0,\dots,x_n) \in K^{n+1}$ by $[x_0:\dots:x_n] \in \mathbb{P}^n_K$.

<u>Note</u> $[x_0:\dots:x_n] = [y_0:\dots:y_n]$ if and only if $(x_0,\dots,x_n), (y_0,\dots,y_n) \in K^{n+1}$ are colinear, i.e.

$(x_0,\dots,x_n) = \lambda(y_0,\dots,y_n)$ for some $\lambda \in K^\times$.



$x_0,\dots,x_n$ are called <u>projective coordinates</u> of the point $[x_0:\dots:x_n] \in \mathbb{P}^n_K$.

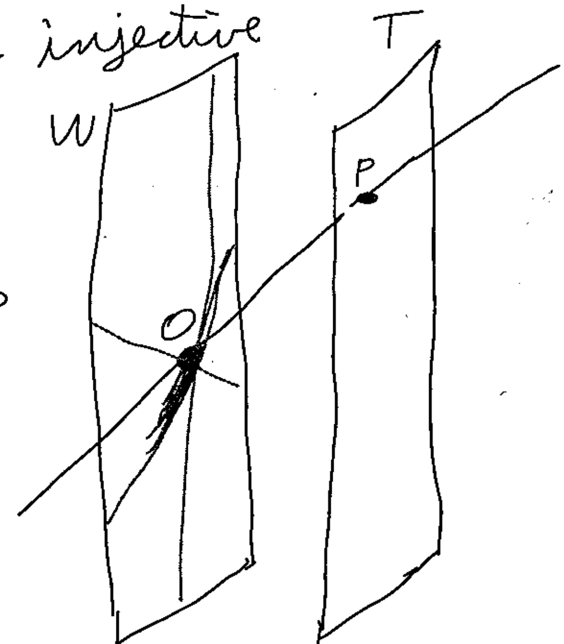
Rmk We could therefore equivalently have defined $\mathbb{P}^n_K$ to be the set of $(n+1)$-tuples $(0,\dots,0)\neq(x_0,\dots,x_n)\in K^{n+1}$ modulo the following equivalence relation:

$$(x_0,\dots,x_n)\sim(y_0,\dots,y_n) \text{ if } (x_0,\dots,x_n)=\lambda(y_0,\dots,y_n) \text{ for some } \lambda\in K^\times.$$

In short: $\mathbb{P}^n_K=\left(K^{n+1}\setminus\{0\}\right)/K^\times$.

Rmk For any n-dimensional affine linear subspace $T\subset K^{n+1}$ not containing the origin, we have an injective map

$$T \hookrightarrow \mathbb{P}^n_K$$
$$P \longmapsto \text{line spanned by } P$$
$$(x_0,\dots,x_n)\mapsto[x_0:\dots:x_n]$$



Its image $U\subset\mathbb{P}^n_K$ (consisting of the lines in K^{n+1} intersecting T) is an affine patch of $\mathbb{P}^n_K$.

For a choice of linear bijection $T \cong K^n$, we obtain a bijection between

$\mathbb{A}^n_K = K^n$ and U called a

<u>chart (map)</u> of $\mathbb{P}^n_K$.

<u>Ex</u> For $i = 0, \dots, n$, we can take

$$T_i = \{ (x_0, \dots, x_n) \in K^{n+1} \mid x_i = 1 \}$$

and the i-th <u>standard chart (map)</u>

$$\varphi_i : K^n \hookrightarrow \mathbb{P}^n_K$$

$$(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \mapsto [x_0 : \dots : x_{i-1} : 1 : x_{i+1} : \dots : x_n]$$

$$\left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right) \mapsto [x_0 : \dots : x_n]$$

with image $U_i = \{ [x_0 : \dots : x_n] \in \mathbb{P}^n_K \mid x_i \neq 0 \}$.

$$\mathbb{P}^n_K \setminus U_i = \{ [x_0 : \dots : x_n] \mid x_i = 0 \}$$

$$\cong \{ [x_0 : \dots : x_{i-1} : x_{i+1} : \dots : x_n] \} = \mathbb{P}^{n-1}_K.$$

<u>Rmk</u> More generally, the complement of U in $\mathbb{P}^n_K$ consists of the lines in K^{n+1} through 0 that are parallel to T, i.e. that lie in

the n-dimensional linear subspace W of K^{n+1} parallel to T.

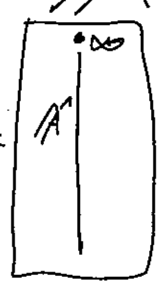
$\Rightarrow$ Identifying W with K^n, we obtain a bijection $\mathbb{P}^n_K \setminus \underbrace{U}_{\mathbb{A}^n_K} \cong (\text{lines through } 0 \text{ in } K^n) \cong \mathbb{P}^{n-1}_K$

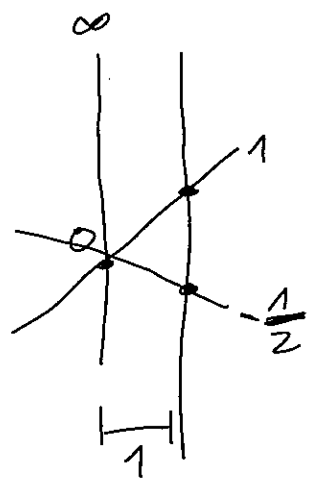
$$\rightsquigarrow \quad \mathbb{P}^n_K = \mathbb{A}^n_K \sqcup \underbrace{\mathbb{P}^{n-1}_K}_{\text{"set of points at } \infty \text{"}}$$

$\underline{\text{Ex}}$ $\quad \mathbb{P}^0_K = \{\text{lines through } 0 \text{ in } K^1\} = \{*\}$

single pt.

$$\mathbb{P}^1_K = \mathbb{A}^1_K \sqcup \{\infty\}$$

$\mathbb{P}^1_K =$ 



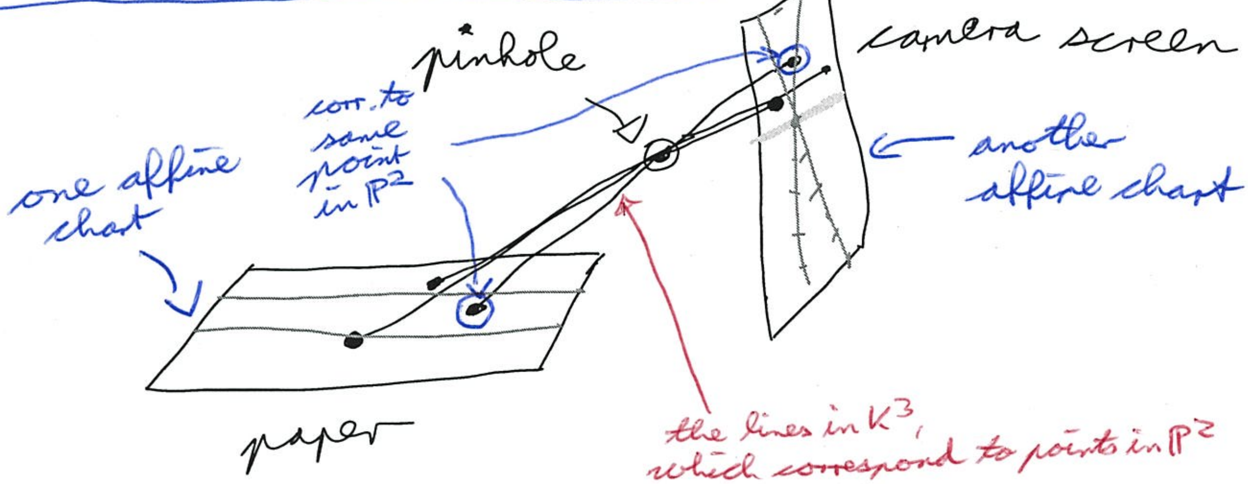
$$\mathbb{P}^2_K = \mathbb{A}^2_K \sqcup \underbrace{\mathbb{P}^1_K}_{\text{pts at } \infty}$$

Rmk The standard affine patches $U_0, \dots, U_n$ cover $\mathbb{P}^n_K$: $\quad \mathbb{P}^n_K = \bigcup_{i=0}^{n} U_i$.

Pf $U_i = \{[x_0 : \dots : x_n] \mid x_i \neq 0\}$.

$\Rightarrow \bigcup_{i=0}^{n} U_i = \{[x_0 : \dots : x_n] \mid x_i \neq 0 \text{ for some } i\}$

But there is (by def.) no point $[x_0 : \dots : x_n] \in \mathbb{P}^n_K$ with $x_0 = \dots = x_n = 0$. $\square$



Def A d-dimensional <u>linear subspace</u> L of $\mathbb{P}^n_K$ is the set of lines through 0 contained in a fixed $(d+1)$-dimensional linear subspace V of K^{n+1}.

Rmk Identifying V with K^{d+1}, we obtain a bijection $L \cong \mathbb{P}^d_K$.

Ex 0-dim. lin. subsp. of $\mathbb{P}^n$
$= $ single point in $\mathbb{P}^n_K$

Ex 1-dim. lin. subsp. are called <u>lines</u> in $\mathbb{P}^n_K$.

Ex 2 <u>planes</u>

Ex $(n-1)$ <u>hyperplanes</u>

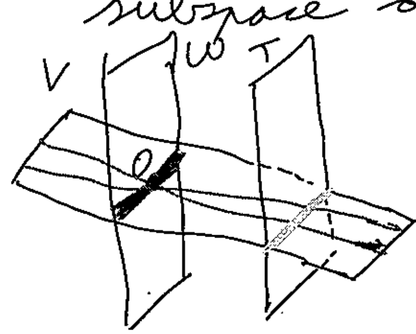
($\mathbb{P}^n_K \setminus U$ as above is a hyperplane in $\mathbb{P}^n_K$.)

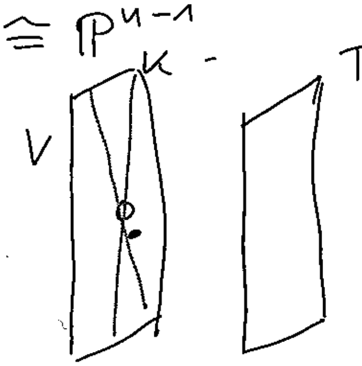
Ex (-1)-dim. lin. subsp. $= \emptyset$.

<u>Lemma [illegible].1.1</u> Let $\varphi: K^n \xrightarrow{\sim} U \subset \mathbb{P}^n_K$ be an affine chart and let $L \subseteq \mathbb{P}^n_K$ be a d-dimensional linear subspace. Then, <u>either</u>

a) $\varphi^{-1}(L) \subseteq K^n$ is an affine d-dimensional linear subspace and $L \cap (\mathbb{P}^n_K \setminus U)$ is a $(d-1)$-dimensional linear subspace of $\mathbb{P}^n_K \setminus U \cong \mathbb{P}^{n-1}_K$.

<u>or</u> b) $\varphi^{-1}(L) = \emptyset$ and L is a d-dimensional linear subspace of $\mathbb{P}^n_K \setminus U \cong \mathbb{P}^{n-1}_K$.

a) 

b) 

Pf Let T be an affine lin. subsp. of K^{n+1} corr. to the affine chart φ. Let W be the linear subspace of K^{n+1} containing 0 parallel to T. It corresponds to $\mathbb{P}^n \setminus U$.
Let V be the lin. subsp. of K^{n+1} corr. to L.

$\dim(T) = \dim(W) = n$

$\dim(V) = d+1$.

Either, a) $V \not\subseteq W$. $\Rightarrow \dim(V \cap W) = d$, so $L \cap (\mathbb{P}^n \setminus U)$ is a $(d-1)$-dimensional linear subspace (consisting of the lines contained in $V \cap W$)

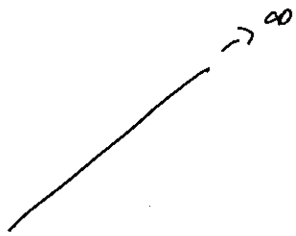
and $\dim(V \cap T) = d$, so $\varphi^{-1}(L) \cong V \cap T$ is an affine d-dimensional linear subspace of $K^n \cong T$.

Or, b) $V \subseteq W \Rightarrow L \subseteq \mathbb{P}^n \setminus U$, $\varphi^{-1}(L) = \emptyset$

$\square$

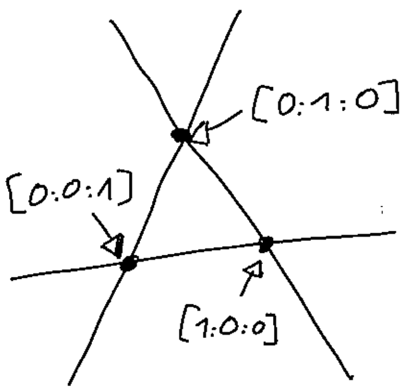
Ex The lines in $\mathbb{P}^2_K = \mathbb{A}^2_K \sqcup \mathbb{P}^1_K$ "are"

- The lines in $\mathbb{A}^2$ (with one point at ∞ each)



- The line $\mathbb{P}^1_K$ at ∞.

$\mathbb{P}^2 \setminus U_0 = \{[x_0 : x_1 : x_2] \mid x_0 = 0\}$



$\mathbb{P}^2 \setminus U_1 = \{[x_0 : x_1 : x_2] \mid x_1 = 0\}$

$\mathbb{P}^2 \setminus U_2 = \{[x_0 : x_1 : x_2] \mid x_2 = 0\}$