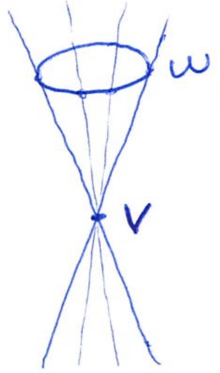
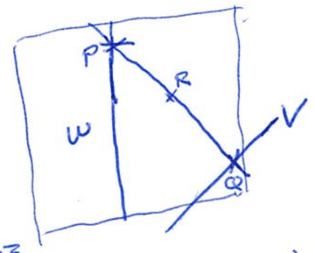
13.3. Applications, part 1

Thm 13.3.1 Let $V, W \subseteq K^n$ be irred. of dimensions a, b.
Let $S \subseteq K^n$ be the union of all straight lines $L \subseteq K^n$ joining a pt. $P \in V$ and a pt. $Q \in W$ with $P \neq Q$.
(S is called the join of V and W.)
If $a+b+2 \leq n$, then $\overline{S} \subsetneq K^n$.
In fact, $\dim(\overline{S}) \leq a+b+1$.

Ex $a=1, b=0, n=3$
V=pt., W=circle $\rightsquigarrow S$ = cone (2-dimensional)



Ex $a=1, b=1, n=3$
V, W skew lines
(non-intersecting, non-parallel)



$\rightsquigarrow S = (K^3 \setminus (A \cup B)) \cup V \cup W \overset{\overline{S}=K^3}{\Rightarrow}$ (3-dimensional),
where A = plane containing V parallel to W,
B = — " — W — " — V

Ex $a=1, b=1, n=3$
V, W parallel lines
$\rightsquigarrow S$ = plane containing V, W (2-dimensional)

Pf of Thm

Consider the morphism

$$\varphi: V \times W \times K \longrightarrow K^n$$
$$(P,\ Q,\ t) \longmapsto \underbrace{tP + (1-t)Q}_{\substack{\text{parametrization}\\ \text{of the line } PQ\\ (\text{if } P \neq Q)}}$$

Its image contains S. (It's equal to S unless $V = W = pt$.)

$$\Rightarrow \dim(\overline{S}) \leq \dim\left(\overline{\varphi(V \times W \times K)}\right)$$
$$\underset{\text{Lemma 10.9}}{\leq} \dim(V \times W \times K)$$
$$= \dim(V) + \dim(W) + \dim(K)$$
$$= a + b + 1 < n.$$

$\square$

Thm 13.3.2 let $d \geq 0$. For any m pts. $P_1, \dots, P_m \in K^n$, if $m < \binom{d+n}{n}$, there is a pol. $0 \neq f \in K[X_1, \dots, X_n]$ of degree $\leq d$ with $P_1, \dots, P_m \in \mathcal{V}(f)$.

Ex $d = n = 2$, $m = 5$

There is a conic (or union of two lines) containing $P_1, \dots, P_5 \in K^2$.

Pf let F_d be the vector space of pol. of deg. $\leq d$.
Consider the linear map

$$\varepsilon : F_d \longrightarrow K^m$$
$$f \longmapsto (f(P_1), \dots, f(P_m))$$

The claim is that ε is not injective.

This follows because

$$\dim(F_d) = \#(\text{monomials } \underset{\substack{\shortparallel \\ X_1^{e_1} \cdots X_n^{e_n}}}{M} \text{ of deg.} \leq d)$$

$$= \#\{(e_1, \dots, e_n) : e_1, \dots, e_n \geq 0,\ e_1 + \dots + e_n \leq d\}$$

$$\overset{\uparrow}{=} \binom{d+n}{n} > m = \dim(K^m).$$

$(e_1, \dots, e_n)$ can be encoded as the string

$$\underbrace{0 \cdots 0}_{e_1} | \underbrace{0 \cdots 0}_{e_2} | \cdots | \underbrace{0 \cdots 0}_{e_n} | \underbrace{0 \cdots 0}_{d-(e_1+\dots+e_n)}$$

consisting of d times the character '0' and n times the character '|'.

$\square$

Rmk In the pf., we only used vector space dimensions. ⇝ The Thm. holds over arbitrary fields!

Thm 13.3.3 For any points $P_1, \dots, P_m \in K^2$, there is an irreducible pol. $0 \neq f \in K[X,Y]$ of degree $\leq m+2$ with $P_1, \dots, P_m \in \mathcal{V}(f)$.

Pf The kernel T of $\varepsilon : F_{m+2} \to K^m$, $f \mapsto (f(P_1), \dots, f(P_m))$

has dimension $\dim(T) \geq \dim(F_{m+2}) - m = \binom{m+4}{2} - m = \frac{m^2+5m+12}{2}$.

It suffices to show that the set

$V := \cancel{\{f \in F_{m+2} \text{ not irreducible}\}}$

$= \{0\} \cup \{f \in F_{m+2} \text{ reducible}\}$

satisfies $\dim(\overline{V}) < \dim(T)$ since we then can't have $T \subseteq V$. Any reducible f of $\deg \leq m+2$ can be written as $g \cdot h$ with $\deg(g) + \deg(h) \leq m+2$.

$$\Rightarrow V = \{0\} \cup \bigcup_{\substack{a,b \geq 1: \\ a+b = m+2}} F_a \cdot F_b$$

In fact, we can make one coeff. of h (say the "leading coefficient") equal to 1.

Let $F_d^1 = \{ f \in F_d$ with at least one coeff. equal to $1 \}$.

(This is an alg. subset of F_d,

$\dim(F_d^1) = \dim(F_d) - 1$.)

$$\Rightarrow V = \{0\} \cup \bigcup_{\substack{a,b \geq 1: \\ a+b \leq m+2}} F_a \cdot F_b^1 .$$

Now, $F_a \cdot F_b^1$ is the image of the morphism

$$F_a \times F_b^1 \longrightarrow F_{m+2} ,$$
$$(g, h) \longmapsto gh$$

so
$$\dim\left(\overline{F_a \cdot F_b^1}\right) \leq \dim(F_a) + \dim(F_b^1)$$
$$= \binom{a+2}{2} + \binom{b+2}{2} - 1$$
$$= \frac{(a+2)(a+1)}{2} + \frac{(b+2)(b+1)}{2} - 1$$
$$= \frac{(a^2+b^2) + 3(a+b) + 2}{2}$$
$$= \frac{(a+b)^2 + 3(a+b) + 2 - 2ab}{2}$$
$$= \frac{(m+2)(m+5)}{2} + 1 - ab$$

$a+b=m+2$

$$\leq \frac{(m+2)(m+5)}{2} + 1 - 1\cdot(m+1)$$
$$= \frac{m^2+5m+10}{2} < \frac{m^2+5m+12}{2} \leq \dim(T).$$

$\Rightarrow \dim(\overline{V}) < \dim(T)$. $\square$

Rmk There's some room for improvement:

~~We only~~

If $f = gh \in T \cap V$, then ($P_1, \dots, P_m \in \mathcal{V}(f) = \mathcal{V}(g) \cup \mathcal{V}(h)$, so) there is a subset $S \subseteq \{1, \dots, m\}$ such that $\{P_i \mid i \in S\} \subseteq \mathcal{V}(g)$ and

$$\{P_i \mid i \notin S\} \subseteq \mathcal{V}(h).$$

$\rightsquigarrow$ can replace F_a, F'_b by

$$F_{a,S} := \{g \in F_a : \forall i \in S : g(P_i) = 0\},$$

$$F'_{b,S} := \{h \in F'_b : \forall i \notin S : h(P_i) = 0\}.$$

(slightly smaller dimension.)

However, the degree can't be improved very much:

Rmk For any $m \geq 2$, there are pts. $P_1, \dots, P_m \in K^2$ s.t. there is no irred. $0 \neq f \in K[X,Y]$ of degree $\leq m-2$ with $P_1, \dots, P_m \in \mathcal{V}(f)$.

Pf Take $P_1, \dots, P_{m-1}$ on the x-axis, P_m not on the x-axis.

The restriction $f(X,0)$ of f to the x-axis is a pol. of ~~deg.~~ $\leq m-2$ vanishing at $m-1$ points $\Rightarrow f(X,0) = 0$

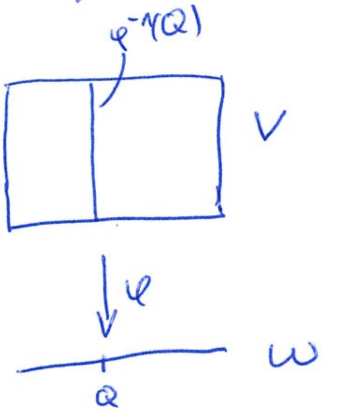
$\Rightarrow Y \mid f(X,Y)$. $\xRightarrow{f \text{ irreducible}} f(X,Y) = c \cdot Y$ for some constant $c \in K^*$.

~~Bet~~

$\Rightarrow f(P_m) \neq 0$. ↯ $\square$

13.4. Dimensions of fibers

Def A fiber of $\varphi: V \to W$ is the preimage $\varphi^{-1}(Q)$ of a point $Q \in W$.



Thm 13.4.1 Let V, W be irred, $\varphi: V \to W$ a morphism, and A an irred. comp. of $\varphi^{-1}(Q)$ for some pt. $Q \in W$. Then, $\operatorname{codim}(A, V) \leq \dim(W)$.

($\Leftrightarrow \dim(A) \geq \dim(V) - \dim(W)$.)

Rmk Compare this to linear maps from linear algebra!

Ex
$$\varphi: K^2 \longrightarrow K^2$$
$$(x, y) \longmapsto (x, xy)$$

$\varphi^{-1}(a, b) = \{(a, \frac{b}{a})\}$ if $a \neq 0$

$\varphi^{-1}(0, b) = \emptyset$ if $b \neq 0$ (fiber can be empty)

$\varphi^{-1}(0, 0) = \{(0, y) \mid y \in K\}$ (fiber can have larger dimension)

We'll prove sth. more general:

Thm 13.4.2 Let V, W, φ as above, $B \subseteq W$ irred.,

A an irred. comp. of $\varphi^{-1}(B)$ with $\overline{\varphi(A)} = B$.

Then, $\operatorname{codim}(A, V) \leq \operatorname{codim}(B, W)$.

Rmk $\overline{\varphi(A)} = B$ is automatic if $B = \{Q\}$.

Rmk In general, the condition $\overline{\varphi(A)} = B$ can't be omitted, as the 2nd example in section 12 (right before the def. of normal alg. sets) shows.

Idea of pf B is almost def. by $r := \operatorname{codim}(B, W)$ equations. $\Rightarrow A$ is almost def. by r equations.

Pf of Thm Let $r = \operatorname{codim}(B, W)$.

By Cor 13.2.2, there are fcts. $g_1, \dots, g_r \in \Gamma(W)$ s.t.

B is an irred. comp. of $V_W(g_1, \dots, g_r)$.

$$\Rightarrow A \subseteq \varphi^{-1}(B) \subseteq \varphi^{-1}(V_W(g_1, \dots, g_r)) = V_V(\underbrace{\varphi^*(g_1)}_{f_1}, \dots, \underbrace{\varphi^*(g_r)}_{f_r})$$

~~A is an~~ Let A' be an irred. comp. of $V_V(f_1, \dots, f_r)$ containing A.

$$\Rightarrow \underbrace{B}_{\text{irred. comp. of } V_W(g_1,\dots,g_r)} = \overline{\varphi(A)} \subseteq \underbrace{\overline{\varphi(A')}}_{\text{irred.}} \subseteq V_W(g_1, \dots, g_r)$$

$$\Rightarrow B = \overline{\varphi(A)} = \overline{\varphi(A')}$$

$$\Rightarrow \underbrace{A}_{\text{irred. comp. of } \varphi^{-1}(B)} \subseteq \underbrace{A'}_{\text{irred.}} \subseteq \varphi^{-1}(B)$$

$\Rightarrow A = A'$, which is an irred. comp. of $V_V(f_1, \dots, f_r)$

$\underset{\text{Thm 13.2.6}}{\Rightarrow}$ $\operatorname{codim}(A, V) \leq r$. $\square$

If φ is dominant, we have equality ~~...~~ in Thm 13.4.1 for a "generic" fiber.

<u>Prop 13.4.4</u> Let V, W, φ as above and assume that φ is dominant. Then, there is ~~a~~ (dense) open subset $\emptyset \neq U \subset W$ ~~...~~ such that ~~every~~ ~~...~~ for every $Q \in U$, ~~...~~ the fiber $\varphi^{-1}(Q)$ is nonempty and every irred. comp. A of $\varphi^{-1}(Q)$ satisfies $\operatorname{codim}(A, V) = \dim(W)$.

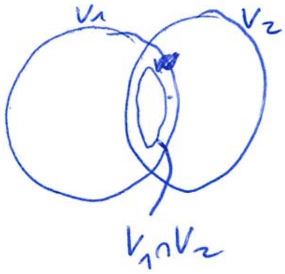
~~I~~ We won't prove this.
(See Thm 1.25 in Shafarevich: Basic Algebraic Geometry 1.)

Wrong Cor 13.4.3

Let $V_1, V_2 \subseteq W$ be irred. and let A be an irred. comp. of $V_1 \cap V_2$. Then,

$$\operatorname{codim}(A, W) \le \operatorname{codim}(V_1, W) + \operatorname{codim}(V_2, W).$$

Ex



Counterexample

$$W = \mathcal{V}(AB - CD) \subset K^4 \qquad (\dim W = 3)$$

$$V_1 = \mathcal{V}(A, C) \subset W \qquad (\dim = 2)$$

$$V_2 = \mathcal{V}(B, D) \subset W \qquad (\dim = 2)$$

$$V_1 \cap V_2 = \mathcal{V}(A, B, C, D) = \{0\} \qquad (\dim = 0)$$

Correct Cor 13.4.3 The above holds if $W = K^n$.

~~Pf shortly~~

(Pf will follow shortly ...)