Cor 13.1.3 Let $V \subseteq W$ both be irreducible.
Then, the codimension

$$\operatorname{codim}(V,W) := \dim(W) - \dim(V)$$

of V in W is the largest $d \geq 0$ s.t. there are irred. alg. sets

$$V = V_0 \subsetneq \dots \subsetneq V_d = W.$$

Pf like Cor 13.1.2. $\square$

Pf of Lemma 13.1.1

Let $n = \dim(W)$. By Noether-Normalization, there is a surjective finite morphism $\varphi : W \to K^n$.

$$V \subsetneq W \overset{\substack{\text{Incomparability} \\ \text{(Lemma 11.6)}}}{\Longrightarrow} \varphi(V) \subsetneq \varphi(W) = K^n$$

Take any $0 \neq f \in K[x_1, \dots, x_n]$ with $\varphi(V) \subseteq \mathcal{V}(f)$.

(Since $\varphi(V)$ is irreducible, we may assume that f is irreducible.)

$\Rightarrow \dim(V) = \dim(\varphi(V)) \leq \dim(\mathcal{V}(f)) = n-1.$

V is contained in an irred. comp. A of $\varphi^{-1}(\mathcal{V}(f))$.

By Thm 12.1 (Going down), we have $\varphi(A) = \mathcal{V}(f)$.

$\Rightarrow \dim(A) = \dim(\varphi(A)) = \dim(\mathcal{V}(f)) = n-1.$ $\square$

13.2. Defining with few equations

Def An irred. $(n-1)$-dimensional irred. subset of K^n is called a *hypersurface* in K^n.

Lemma 13.2.1 Any hypersurface $V \subseteq K^n$ is of the form $V = \mathcal{V}(f)$ for some irred. $0 \neq f \in K[x_1, \ldots, x_n]$.

Pf $V \subsetneq K^n \Rightarrow V \subseteq \mathcal{V}(f)$ for some $0 \neq f$.

V irred. $\Rightarrow V \subseteq$ some irred. comp. of $\mathcal{V}(f)$.

$\Rightarrow$ w.l.o.g. f irreducible

If $V \subsetneq \mathcal{V}(f)$, then $\dim(V) < \dim(\mathcal{V}(f)) = n-1$. ↯

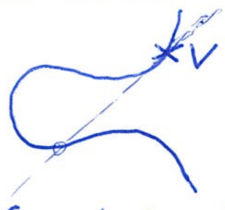
$\Rightarrow V = \mathcal{V}(f)$. $\square$

Cor 13.2.2 Let $V \subseteq W$ both be irred. Then, there are $c := \operatorname{codim}(V, W)$ functions $f_1, \ldots, f_c \in \Gamma(W)$ s.t. V is an irred. comp. of $\mathcal{V}_W(f_1, \ldots, f_c)$ and all other irred. comp. also have codimension c in W.

Rmk Let $K = \mathbb{C}$, $W = \mathcal{V}(X^3 - 4X + 4 - Y^2) \subset K^2$,

$V = \{(2,2)\}$ or $\{(\pi, \sqrt{\pi^3 - 4\pi + 4})\}$ (easier).

Then, there is no $f \in \Gamma(W)$ with $V = \mathcal{V}_W(f)$.



(Of course, there are plenty of f s.t. $\mathcal{V}_W(f)$ is a finite set of pts. including V.)

Pf of Rmk skipped (not easy). "$\square$"

Pf of Cor 13.2.3 by induction over c.

$c=0$: clear ($V=W$).

$c-1 \to c$: Let $n=\dim(W)$, $\varphi: W \to K^n$ a surjective finite morphism.

$\operatorname{codim}(\varphi(V), K^n) = \operatorname{codim}(V, W) = c \geq 1$.

Let $0 \neq g_1 \in K[x_1, \dots, x_n]$ be irreducible with $\varphi(V) \subseteq \mathcal{V}(g_1)$.

$\operatorname{codim}(\varphi(V), \mathcal{V}(g_1)) = c-1$.

By induction, there are $g_2, \dots, g_c \in K[x_1, \dots, x_n]$ s.t. $\varphi(V)$ is an irred. comp. of $\mathcal{V}(g_1, \dots, g_c) = \mathcal{V}_{\mathcal{V}(g_1)}(g_2, \dots, g_c)$ and all other irred. comp. also have codimension $c-1$ in $\mathcal{V}(g_1)$, so codimension c in K^n.

$\Rightarrow$ By Thm 12.1 (going down), the irred. comp. of $\varphi^{-1}(\mathcal{V}(g_1, \dots, g_c)) = \mathcal{V}(\underbrace{\varphi^*(g_1)}_{f_1}, \dots, \underbrace{\varphi^*(g_c)}_{f_c})$

have the same dimension, so have codim. c in W.

$V \subseteq \varphi^{-1}(\mathcal{V}(g_1, \dots, g_c))$ is contained in, and hence equal to (because of dimension), one of them. $\square$

Lemma 13.2.4

Let S be a module-finite ring ext. of R and assume that S, R are int. dom. with fields of fractions M, L.

$$\begin{array}{ccc} M & \supseteq & S \\ | & & | \\ L & \supseteq & R \end{array}$$

Let $a \in S$ and let $b = N_{M/L}(a) \in L$.
If R is integrally closed in L, then $b \in R$ and $a \mid b$ in S.

Pf when M/L is a Galois ~~ext~~ and S is the int. closure of R in M

~~If M/L is Galois ext~~ Then, $b = \prod_{\sigma \in Gal(M/L)} \underbrace{\sigma(a)}_{\in S}$.

This is integral over R because $a \in S$, and therefore each $\sigma(a)$, is. Hence, $b \in R$.
It is divisible by a because a is one of the factors in b.

__Pf in general__

Let $f \in R[X]$ be a monic pol with $f(a)=0$ and let $g \in L[X]$ be the min. pol. of a.

$\Rightarrow g \mid f$.

$\Rightarrow$ Every root α_i of g in M is integral over R.

$\Rightarrow$ Every coeff. of $g(X) = \prod\limits_{\substack{i \\ \text{(with mult.)}}} (X-\alpha_i)$ is integral over R and lies in L. $\Rightarrow$ Every coeff. lies in R: $g \in R[X]$.

Write $g(X) = X^n + c_{n-1}X^{n-1} + \dots + c_0$.

$$\Rightarrow b = N_{M|L}(a) = N_{L(a)|L}(N_{M|L(a)}(a))$$
$$= N_{L(a)|L}(a^{[M:L(a)]})$$
$$= N_{L(a)|L}(a)^{[M:L(a)]}$$
$$= (\pm c_0)^{[M:L(a)]} \in R.$$

Also, $0 = g(a) = a^n + c_{n-1}a^{n-1} + \dots + a c_1 + c_0$.

$$= a(\underbrace{a^{n-1} + c_{n-1}a^{n-2} + \dots + c_1}_{\in S}) + c_0.$$

$\Rightarrow a \mid c_0 \mid b$ in S. $\square$

Thm 13.2.5 (Krull's principal ideal theorem)

Let W be an irred. alg. set and let V be an irred. comp. of $\mathcal{V}(f) \subseteq W$ for some $0 \neq f \in \Gamma(W)$.

Then, $\operatorname{codim}(V, W) = 1$. ($\Leftrightarrow \dim(V) = \dim(W) - 1$).

Pf Let $n = \dim(W)$, $\varphi : W \to K^n$ a surj. fin. morphism.

$$\begin{array}{cccccc} W & & K(W) & \supseteq & \Gamma(W) & \ni f \\ \varphi \big\downarrow & & \big| & & \big| & \\ K^n & & K(x_1, \dots, x_n) & \supseteq & K[x_1, \dots, x_n] & \end{array}$$

Let $g = \mathrm{Nm}_{K(W)|K(x_1,\dots,x_n)}(f)$. Clearly, $g \neq 0$.

By Lemma 13.2.4, we have $g \in K[x_1, \dots, x_n]$,

$\varphi^*(g) \mid f$ in $\Gamma(W)$.

$\Rightarrow V \subseteq \mathcal{V}_W(f) \subseteq \mathcal{V}_W(\varphi^*(g)) = \varphi^{-1}(\mathcal{V}_{K^n}(g))$.

$\Rightarrow \varphi(V) \subseteq \varphi(\mathcal{V}_W(f)) \subseteq \mathcal{V}_{K^n}(g)$

It suffices to show that $\varphi(\mathcal{V}_W(f)) = \mathcal{V}_{K^n}(g)$: Then, by Thm 12.1 (going down), $\dim(V) = \dim(\varphi(V)) = \dim(\mathcal{V}_{K^n}(g)) = n-1$.

Assume that $\varphi(\mathcal{V}_W(f)) \subsetneq \mathcal{V}_{K^n}(g)$.

Take $0 \neq h \in K[x_1, \dots, x_n]$ with

$h|_{\varphi(\mathcal{V}_W(f))} = 0$, but $h|_{\mathcal{V}_{K^n}(g)} \neq 0$.

$$h|_{\varphi(\mathcal{V}_W(f))} = 0 \Rightarrow \varphi^*(h)|_{\mathcal{V}(f)} = 0$$

$$\underset{\text{Nullstellensatz}}{\Rightarrow} \quad \varphi^*(h)^m \in \mathcal{I}_W(\mathcal{V}_W(f)) = \sqrt{(f)} \subseteq \Gamma(W)$$

for some $m \geq 1$.

$$\Rightarrow \varphi^*(h)^m = fe \text{ for some } e \in \Gamma(W).$$

$$\Rightarrow \mathrm{Nm}(\varphi^*(h))^m = \underbrace{\mathrm{Nm}(f)}_{g}\,\underbrace{\mathrm{Nm}(e)}_{\in K[x_1,\dots,x_n]}$$

$\| \leftarrow h \in K[x_1,\dots,x_n]$

$h^{m\cdot[K(W):K(x_1,\dots,x_n)]}$

$$\Rightarrow h^{\cdots} \in (g) \subseteq K[x_1,\dots,x_n]$$

$$\Rightarrow h \in \sqrt{(g)} = \mathcal{I}(\mathcal{V}(g))$$

$$\Rightarrow h|_{\mathcal{V}(g)} = 0 \quad \lightning$$

$\square$

Rmk ~~$\mathcal{V}(f_1,\dots,f_r)$ can be empty, even if $r < \dim(W)$.~~
$\mathcal{V}(f)$ can be empty.

Rmk The Thm can fails if K isn't alg. closed:
$\mathcal{V}(x^2+y^2) \subseteq \mathbb{R}^2$ has codimension 2.
$\mathcal{V}(x^2+y^2) = \{(0,0)\}$

Cor 13.2.6 Let W be an irred. alg. set and let V be an irred. comp. of $\mathcal{V}_W(f_1, \dots, f_r)$ for some $f_1, \dots, f_r \in \Gamma(W)$.

Then, $\operatorname{codim}(V, W) \leq r$.

Pf (by induction over r): Assume $r \geq 1$. Let A be an irred. comp. of $\mathcal{V}_W(f_1)$ containing V.

$$\underset{\text{Thm 13.2.5}}{\Longrightarrow} \operatorname{codim}(A, W) \leq 1.$$

V is an irred. comp. of $\mathcal{V}_A(f_2, \dots, f_r)$.

By ind., $\operatorname{codim}(V, A) \leq r - 1$.

$\Longrightarrow \operatorname{codim}(V, W) \leq r$. $\square$

Rmk Of course, we can have $\operatorname{codim}(V, W) < r$:
(E.g. could have $f_1^3 = f_2 + f_3^2$. $\rightsquigarrow$ f_1 redundant.)