Ese $K=\mathbb{C}$, $V=\mathcal{V}(x^2+y^2-1)$, $W=K$

$$\varphi : V \longrightarrow W$$
$$(x,y) \longmapsto x$$

$\Gamma(W)=K[x]$, $K(W)=K(x)$

$$\begin{array}{c} \\ \downarrow \varphi^* \\ K(W) = K(x)(\sqrt{1-x^2}) \end{array}$$

$\Gamma(V)=K[x,y]/(x^2+y^2-1)$,

$$\|$$

$K[x][\sqrt{1-x^2}]$

$Gal(K(x)(\sqrt{1-x^2}) \mid K(x)) = \{id, \sigma\}$, where the aut σ given by $\sigma(\sqrt{1-x^2}) = -\sqrt{1-x^2}$ (i.e. $\sigma(y)=-y$), corr. to the reflection across the x-axis.

Ex $V = K^n$, $W = K^n$

$$\varphi : V \longrightarrow W$$
$$(a_1, \ldots, a_n) \longmapsto (b_0, \ldots, b_{n-1}),$$
where $\prod_{i=1}^{n}(x - a_i) = x^n + b_{n-1}x^{n-1} + \ldots + b_0$

($b_i = \pm$ elem. symm. pol. in $a_1, \ldots, a_n$)

i-th

φ is surjective.

φ is finite because $A_i^n + \varphi^*(B_{n-1})A_i^{n-1} + \ldots + \varphi^*(B_0) = 0$.

$K(V)$ is a Galois ext. of $\varphi^*(K(W)) = K($elem. symm. pol.$)$

$\parallel$

$K(A_1, \ldots, A_n)$

with Galois group S_n.

S_n acts on V by permuting $a_1, \ldots, a_n$ and this action is by deck transformations (= leaves $b_0, \ldots, b_{n-1}$ unchanged).

$K(V) = K(B_1, \ldots, B_n)$(roots of $X^n + B_{n-1}X^{n-1} + \ldots + B_0$ in $\overline{K(B_1, \ldots, B_n)}$).

"$W = V/S_n$" (Geometric Invariant Theory)

Thm 12.2 Let V, W, φ as in the Remark and let $B \subseteq W$ be irred. Then, $G := Gal(\ldots)$ acts transitively on the irred. comp. $A_1, \ldots, A_r$ of $\varphi^{-1}(B)$.

~~In part, the irred. comp.~~

Cor 12.3 The irred. comp. have the same image:
$\varphi(A_1) = \ldots = \varphi(A_r) = B$.

~~B) dim(A_1) = ... dim~~

Pf of Cor If $\varphi(A_1) = B$, $\alpha_\sigma(A_1) = A_i$, then $\varphi(A_i) = \varphi(\alpha_\sigma(A_1)) = \varphi(A_1) = B$. $\square$

Pf of Thm ~~Assume w.l.o.g. that $A_1, \ldots, A_k$~~

Assume w.l.o.g. that $A_1, \ldots, A_k$ form a G-orbit, $\varphi(A_1) = B$, and $1 \leq k < r$.

Pick points $P_{k+1} \in A_{k+1} \setminus A_1, \ldots, P_r \in A_r \setminus A_1$.

By Cor 6.2, there is a function $f \in \Gamma(V)$ with

$$f|_{A_1} = 0 \text{ and } f(P_{k+1}) = \ldots = f(P_r) = 1.$$

$$\Downarrow$$

$$f|_{A_{k+1}}, \ldots, f|_{A_r} \neq 0$$

$$\Downarrow \; G \text{ permutes } A_{k+1}, \ldots, A_r$$

$$\sigma(f)|_{A_{k+1}}, \ldots, \sigma(f)|_{A_r} \neq 0 \quad \forall \sigma \in G$$

$$\Downarrow \; \Gamma(A_{k+1}), \ldots, \Gamma(A_r) \text{ int. dom. because } A_{k+1}, \ldots, A_r \text{ are irred.}$$

$$g|_{A_1} = 0 \qquad g|_{A_{k+1}}, \ldots, g|_{A_r} \neq 0$$

for $g := Nm_{K(V)|K(W)}(f) = \left(\prod_{\sigma \in G} \sigma(f)\right)^t$, where $t \geq 1$ is the degree of inseparability of $K(V) | \varphi^*(K(W))$.

~~But $g \in \varphi^*(K(W))$~~ [illegible]

~~is the composition of a rat. fct.~~

We have $g \in \varphi^*(K(W)) \cap \Gamma(V)$

$$= \varphi^*(\Gamma(W)).$$

$$\uparrow$$

$$\text{LHS} = \{ h \in \varphi^*(K(W)) \text{ int. over } \varphi^*(\Gamma(W)) \}$$

$$= \{ \varphi^*(h) : h \in K(W) \text{ int. over } \Gamma(W) \}$$

$$= \varphi^*(\Gamma(W))$$

$$\uparrow$$

$\Gamma(W)$ int. cl. in $K(W)$

Let $g = \varphi^*(h)$, $h \in \Gamma(W)$.

Now, $\underset{\substack{\parallel \\ h \circ \varphi|_{A_1}}}{g|_{A_1}} = 0$ with $\varphi(A_1) = B$ implies $h|_B = 0$.

$\Rightarrow g|_{A_i} = 0 \ \forall i.$ ↯ $\square$

We neglected one thing in the pf of Thm 12.1:

<u>Lemma 12.4</u>

Let R be an int. dom. with field of fractions F.

Let $L|F$ be a finite field ext.

~~If R is fin. gen. as a ring ext. of K,~~

Assume that R is a finitely generated ring ext. of K and integrally closed in F.

Then, the integral closure S of R in L is a) a fin. gen. ring ext. of K and b) a fin. gen. R-module.

<u>Rmk</u> It suffices to show b).

Since $R=K[X_1,\dots]/\dots$ is noetherian, submod. of fin. gen. R-mod. are fin. gen. $\implies$

<u>Pf when $L|F$ is separable</u> (e.g. if $\mathrm{char}(K)=0$)

We can replace L by its normal (= Galois) closure over F.

Let $n=[L:F]$. Pick basis $b_1,\dots,b_n$ of $L|F$.

Multiplying by elements of R, we can make $b_1,\dots,b_n\in S$ according to Lemma 4.5.2.

Let $c=\sum_{i=1}^n a_i b_i \in S$ with $a_1,\dots,a_n\in F$.

Note: For any $x\in S$, $\underset{\sum_{\sigma\in \mathrm{Gal}(L|F)}\sigma(x)}{\underset{\|}{\mathrm{Tr}(x)}}\in F$ is integral over R, so $\mathrm{Tr}(x)\in R$

Now, $\mathrm{Tr}(b_j c)=\sum_{i=1}^n a_i\,\mathrm{Tr}(b_i b_j)$ for all j.

~~This is a system of~~

$$\Longrightarrow \underbrace{(\mathrm{Tr}(b_j c))_{j=1,\dots,n}}_{\in R^n\subseteq F^n} = \underbrace{(\mathrm{Tr}(b_i b_j))_{i,j=1,\dots,n}}_{n\times n\text{-matrix with coeff. in } R\subseteq F} \cdot \underbrace{(a_i)_{i=1,\dots,n}}_{\in R^n\subseteq F^n} \qquad (I)$$

The matrix $M = (\mathrm{Tr}(b_i b_j))_{i,j}$ is invertible because $L|F$ is separable. $\Rightarrow \det(M) \neq 0$.

Also, $(*)$ implies that
$\det(M) \cdot a_i \in R$ for $i = 1, \dots, n$.

$$\Rightarrow S \subseteq \underbrace{\frac{1}{\det(M)} \cdot (b_1 R + \dots + b_n R)}_{\text{fin.-gen. } R\text{-mod.}}$$

$\Rightarrow S$ is a fin. gen. R-mod. $\square$

Pf for arbitrary $L|F$

By Problem 4a on Pset 7, there are alg. indep. elements $x_1, \dots, x_n$ of R s.t. R is an int. ext. of $K[x_1, \dots, x_n]$.

Replacing R by $K[x_1, \dots, x_n]$, F by $K(x_1, \dots, x_n)$, we can assume $R = K[x_1, \dots, x_n]$, $F = K(x_1, \dots, x_n)$.

Let $F \subseteq M \subseteq L$ be the largest purely inseparable subext. of $L|F$. Let $p = \mathrm{char}(K) \neq 0$.

$\Rightarrow M \subseteq K(x_1^{1/p^t}, \dots, x_n^{1/p^t})$ for some $t \geq 0$.
and $L|M$ is separable.

The int. closure of $R = K[x_1, \dots, x_n]$ in $K(x_1^{1/p^t}, \dots, x_n^{1/p^t})$ is $K[x_1^{1/p^t}, \dots, x_n^{1/p^t}]$ (Why?) and hence a fin. gen. $K[x_1, \dots, x_n]$-module. $\Rightarrow$ The int. closure of R in M is a fin. gen. R-module. The result then follows from the separable case since $L|M$ is separable. $\square$

13. More about dimension

13.1. Another definition of dimension

Lemma 13.1.1 Let $V \subsetneq W$ be irreducible alg. sets.

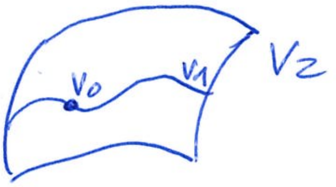
Then, a) $\dim(V) < \dim(W)$

b) There is an alg. set $V \subseteq A \subsetneq W$ with

$$\dim(A) = \dim(W) - 1.$$

Cor 13.2 Let $V \neq \emptyset$ be any alg. set. Then, $\dim(V)$ is the largest $d \geq 0$ s.t. there are irreducible alg. sets

$$\emptyset \neq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_d \subseteq V.$$



Pf $\dim(V) \geq d$:

$$0 \leq \dim(V_0) < \dim(V_1) < \dots < \dim(V_d) \leq \dim(V).$$

$\dim(V) \leq d$: Let $d' = \dim(V)$.

Let $V_{d'}$ be an irred. comp. of V of dimension $d' = \dim(V)$.

If $d' \geq 1$, let $P \in V_{d'}$, so $\{P\} \subsetneq V_{d'}$.

There is an irred. subset $\{P\} \subseteq V_{d'-1} \subsetneq V_{d'}$ of dimension $d'-1$.

$\vdots$

$\leadsto \quad V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_{d'} \subseteq V.$ $\square$

Rmk The cor ~~[illegible]~~ fails without the assumption that $V_0, \dots, V_d$ are irreducible:

If $P_1, \dots, P_n$ are points in V, then

$$\{P_1\} \subsetneq \{P_1, P_2\} \subsetneq \dots \subsetneq \{P_1, \dots, P_n\} \subseteq V,$$

so there are arbitrarily long chains if $|V| = \infty$.