12. Going down theorem

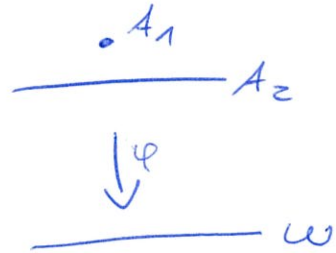
<u>Reminder</u> Let $\varphi: V \to W$ be a surjective finite morphism and let B be an irred. subset of W.
Decompose $\varphi^{-1}(B)$ into irred. comp.:

$$\varphi^{-1}(B) = A_1 \cup \dots \cup A_r.$$

Then, $\varphi(A_i) = B$ for <u>some</u> component A_i

<u>Rmk</u> We might not have $\varphi(A_i) = B$ for <u>all</u> components A_i.

<u>Ex</u> $V = \mathcal{V}(y) \cup \{(0,1)\}$



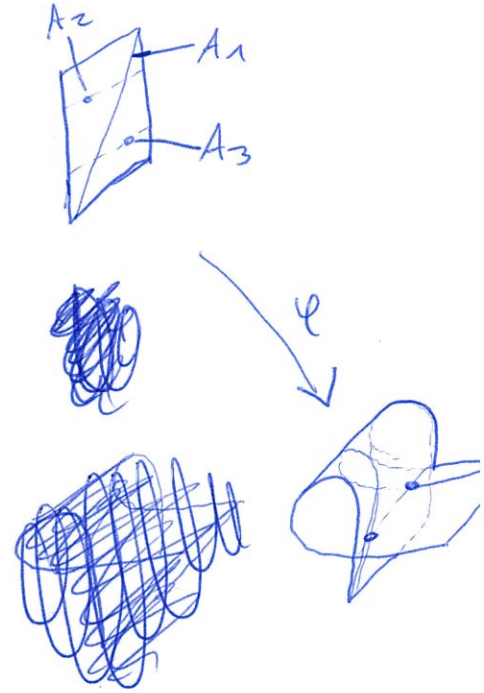
$W = K$

$\varphi: V \to K$ is finite because its restrictions to A_1, A_2 are.
$(x,y) \mapsto x$

$$V = \varphi^{-1}(W) = \underbrace{A_1}_{\substack{\mathrm{im}=\{0\}\\ \neq W}} \cup \underbrace{A_2}_{\mathrm{im}=W}$$

Cause: V not irreducible

Ese $V = K^2$



$$W = \mathcal{V}(x^2(x+1) - y^2)$$

$$\varphi : V \longrightarrow W$$

$$(t,u) \longmapsto (t^2-1,\ t(t^2-1),\ u)$$

φ is finite: $T^2 - 1 - \varphi^*(X) = 0$

$$U - \varphi^*(Z) = 0$$

$B = \varphi(\underbrace{\mathcal{V}(T-U)}_{\text{irred.}})$ is irreducible

$$\varphi^{-1}(B) = \underset{\substack{\uparrow \\ \text{im}=B}}{A_1} \cup \underset{\substack{\uparrow \\ \text{im}\neq B}}{A_2} \cup \underset{\substack{\uparrow \\ \text{im}\neq B}}{A_3}$$

Cause: W not normal

Def An irred. alg. set $V \subseteq K^n$ is <u>normal</u> if the ring $\Gamma(V)$ is integrally closed in its field of fractions $K(V)$.

Ex K^n is normal because $K[X_1, \dots, X_n]$ is a UFD.

Thm 12.1 (Going down)

Let V be an <u>irred</u>. alg. set and let W be a <u>normal</u> alg. set. Let $\varphi : V \to W$ be a surj. finite morphism. Let B be an irred. alg. subset of W and decompose $\varphi^{-1}(B)$ into irred. comp.: $\varphi^{-1}(B) = A_1 \cup \dots \cup A_r$.

Then, $\varphi(A_i) = B$ for <u>every</u> component A_i.

$$\begin{tikzcd}
V \arrow[d, two heads] & \supseteq & \varphi^{-1}(B) \arrow[d, two heads] & = & A_1 \arrow[dll, two heads] & \cup \dots \cup & A_r \arrow[dllll, two heads] \\
W & \supseteq & B & & & &
\end{tikzcd}$$

Pf ~~φ* is an inclusion~~

φ dominant $\Rightarrow$ $\overset{\text{injections}}{\varphi^*} : \Gamma(W) \hookrightarrow \Gamma(V)$

$\varphi^* : K(W) \hookrightarrow K(V)$

We'll consider $\Gamma(W), K(W)$ subsets of $\Gamma(V), K(V)$ via these injections.

W normal: $\Gamma(W)$ integrally closed in $K(W)$

φ finite: $\Gamma(V)$ integral ring ext. of $\Gamma(W)$

$K(V)$ alg. field ext. of $K(W)$

Let L be the normal closure of this field ext. It is a finite alg. ext. of $K(W)$ containing $K(V)$.

Let S be the integral closure of $\Gamma(W)$ in L.

We show in Lemma 12.4 that S is a ring-finite ext. of k.

$\Rightarrow S = \Gamma(V')$ for some alg. set V'.

The inclusion $\Gamma(V) \hookrightarrow \Gamma(V') = S$ corr. to a dominant morphism $\psi : V' \longrightarrow V$.

Since $\Gamma(V') = S$ is an int. ext. of $\Gamma(W)$ and hence of $\Gamma(V)$, the morphism ψ is finite.

Now, decompose $(\varphi\circ\psi)^{-1}(B)$ into irred. comp.:

$$(\varphi\circ\psi)^{-1}(B) = A_1' \cup \dots \cup A_s'.$$

$$\begin{tikzcd}
L & \supseteq & S=\Gamma(V') & & V' \arrow[d, two heads, "\text{fin.}"] & \supseteq & A_1'\cup\dots\cup A_s' \arrow[d] \\
K(V) \arrow[u, hook] & \supseteq & \Gamma(V) \arrow[u, hook] & & V \arrow[d, two heads, "\text{fin.}"] & \supseteq & A_1\cup\dots\cup A_r \arrow[d] \\
K(W) \arrow[u, hook] & \supseteq & \Gamma(W) \arrow[u, hook] & & W & \supseteq & B
\end{tikzcd}$$

<u>Claim</u> Any A_i contains $\psi(A_j')$ for some j.

Pf w.l.o.g. $i=1$. Let $P\in A_1\setminus(A_2\cup\dots\cup A_r)$.

Take any preimage P' in V'. It lies in some irred. comp. A_j'.

$\Rightarrow$ $\psi(A_j') \subseteq A_1\cup\dots\cup A_r$

$\Rightarrow P\in\psi(A_j') \subseteq A_k$ for some k

$\underset{P\notin A_2\cup\dots\cup A_r}{\Rightarrow}$ $k=1$.

$\square$ (claim)

It then suffices to ~~prove the Thm~~ show that $(\varphi\circ\psi)(A_j')=B$ $\forall j$.

$\Rightarrow$ We may assume w.l.o.g. that $K(V)$ is a normal field ext. of $K(W)$ and that $\Gamma(V)$ is the int. closure of $\Gamma(W)$ in $K(V)$. This case will be handled in Cor 12.3 below. $\square$

__Rmk__ Let V, W be normal alg. sets, $\varphi : V \to W$ a dominant finite morphism, $K(V)$ a normal field ext. of $\varphi^*(K(W))$.

Then, any automorphism $\sigma \in Gal(K(V) | \varphi^*(K(W)))$ of $K(V)$ fixing $\varphi^*(K(W))$ restricts to an automorphism of $\Gamma(V)$ fixing $\varphi^*(\Gamma(W))$.

This automorphism corresponds to a morphism

- $\psi_\sigma : V \to V$ with $\psi_\sigma^* = \sigma$

with $\varphi\psi_\sigma = \varphi$ because $\underbrace{\psi_\sigma^*}_{\sigma} \circ \varphi^* = \varphi^*$.

(a "__deck transformation__" of $\varphi : V \to W$).

$$\begin{tikzcd} V \arrow[loop right, "\alpha_\sigma"] \arrow[d, "\varphi"] \\ W \end{tikzcd}$$

We obtain an action of $Gal(\ldots)$ on V by

- deck transformations.

Note that this action permutes the irred. comp. of $\varphi^{-1}(B)$ for any alg. subset $B \subseteq W$.