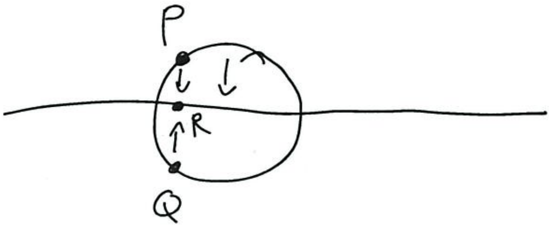
Lemma 11.6 (Incomparability)

Let $\varphi : V \to W$ be a finite morphism and let $V_1 \subsetneq V_2 \subseteq V$ be alg. subsets with V_2 irreducible. Then $\varphi(V_1) \subsetneq \varphi(V_2) \subseteq W$.

<u>Rmk</u> This can fail if V_2 is reducible:



$V_1 = \{P\} \rightsquigarrow \varphi(V_1) = \{R\}$

$V_2 = \{P, Q\} \rightsquigarrow \varphi(V_2) = \{R\}$

<u>Rmk</u> We'll soon show that $V_1 \subsetneq V_2$, V_2 irreducible implies that

$$\begin{array}{ccc} \dim(V_1) & < & \dim(V_2) \\ \| & & \| \\ \dim(\varphi(V_1)) & & \dim(\varphi(V_2)) \end{array}$$

<u>Pf</u> W.l.o.g. $V = V_2$, $\varphi(V) = W$.

Let $0 \neq f \in \Gamma(V)$ with $f|_{V_1} = 0$.

Since $\Gamma(V)$ is an integral ext. of $\varphi^*(\Gamma(W))$, there is a monic polynomial equation

$$f^n + \varphi^*(c_{n-1}) f^{n-1} + \dots + \varphi^*(c_0) = 0 \qquad (I)$$

with $c_{n-1}, \dots, c_0 \in \Gamma(W)$. Pick one of smallest possible degree n.

$f|_{V_1} = 0 \overset{(I)}{\Longrightarrow} \varphi^*(c_0)\big|_{V_1} = 0.$ ~~$-f^n - \varphi^*(c_{n-1})f^{n-1} - \dots - \varphi^*(c_1)f|_{V_1} = 0.$~~

$\Rightarrow \varphi(V_1) \subseteq \mathcal{V}_W(c_0)$.

$\Rightarrow$ If $\varphi(V_1) = W$, then $c_0 = 0$.

$\Rightarrow f^{n-1} + \varphi^*(c_{n-1}) f^{n-2} + \dots + \varphi^*(c_0) = 0$,

contradicting the minimality of n.

$f \neq 0$ and $\Gamma(V)$ is an integral domain because V is irreducible

$\square$

Thm 11.7 (Noether Normalisation)

Let $V \subseteq K^m$ be an irred. alg. set of dimension n.

Then, there is a <u>finite</u> morphism $\pi : V \to K^n$.

In fact, there is a finite linear projection onto K^n.

~~Pf surjectivity follows automatically~~

Rmk Any such morphism is surjective.

Pf of Rmk $\pi(V) \subseteq k^n$ is closed because π is finite, (Cor 11.5).

$\dim(\pi(V)) = \dim(V) = n$ by Thm 11.2.

If $\pi(V) \subseteq \mathcal{V}(f)$ for some $0 \neq f \in k[X_1, \dots, X_n]$, Then

$$\dim(\pi(V)) \underset{\text{Lemma 10.10}}{\leq} \dim(\mathcal{V}(f)) \underset{\text{Thm 10.7}}{=} n-1 \quad ↯.$$

$$\Rightarrow \pi(V) = k^n. \qquad \square$$

~~14. Noether normalization~~

~~Thm 14.1 Let $V \subseteq K^m$ be an irred. alg. set of dimension n.~~

~~Then, there is a [illegible] finite morphism $\pi: V \to K^n$.~~

~~In fact, there is such a projection onto an n-dimensional linear subspace of K^m.~~

~~INSERT(X)~~

Exe The proj. of $V = V(XY-1)$ onto the x-axis (or the y-axis) is dominant, but not surjective (hence not finite). However, the proj.

$$\pi : V \to K \quad \text{is surjective:}$$
$$(x,y) \mapsto x+y$$

The preimages of $t \in K$ are the points (x,y) with $x^2 - tx + 1 = 0$ and $y = t - x$.

In fact, it's finite because

$$X^2 - \pi^*(T)X + 1 = 0$$

and

$$Y^2 - \pi^*(T)Y + 1 = 0$$

in $\Gamma(V)$.

Pf of Thm 11.7 We have $n = \dim(V) \leq \dim(K^m) = m$.

we use induction over $m-n$.

If $n = m$, we're done. (Take $\pi = id$.)

Assume $m \geq n+1$.

~~It suffices to construct a~~

~~We use induction~~

~~Let's construct a fin. mor~~

It suffices to construct a fin. proj. $\pi_1 : \underset{K^m}{\overset{\cap}{V}} \to K^{m-1}$:

$\Rightarrow \dim(\pi(V)) = \dim(V) = n$.

By ind., there is then a fin. proj. $\pi_2 : \pi(V) \to K^n$.

Then we can take $\pi := \pi_2 \circ \pi_1$.

To construct π_1:

Observe that $V \subsetneq K^m$, so there is a pol.

$0 \neq f \in K[X_1, \dots, X_m]$ which vanishes on V:

$f(x_1, \dots, x_m) = 0$ in $\Gamma(V)$.

Consider a proj. $\pi_1 : K^m \longrightarrow K^{m-1}$ of the form

$(a_1, \dots, a_m) \longmapsto (a_1 - c_1 a_m, \dots, a_{m-1} - c_{m-1} a_m)$

with $c_1, \dots, c_{m-1} \in K$.

Its pullback map is

$$\pi_1^* : K[Y_1, \dots, Y_{m-1}] \longrightarrow K[X_1, \dots, X_m]$$

$$Y_i \longmapsto X_i - c_i X_m .$$

In $\Gamma(V)$, we have ~~can write~~

$$0 = f(X_1, \dots, X_m) = f(X_1 - c_1 X_m + c_1 X_m, \dots, X_{m-1} - c_{m-1} X_m + c_{m-1} X_m, X_m)$$

$$= f(\pi_1^*(Y_1) + c_1 X_m, \dots, \pi_1^*(Y_{m-1}) + c_{m-1} X_m, X_m).$$

The RHS is a (after expanding) pol. in X_m with coeff. in $\pi_1^*(\Gamma(K^{m-1}))$.

If the pol. f has degree d, then the RHS has degree $\leq d$.

The X_m^d-coeff. of the right-hand side lies in K.
[and depends on $c_1, \dots, c_{m-1}$.]
It is a nonzero pol. in $c_1, \dots, c_{m-1}$:

If $f = \sum r_{i_1 \cdots i_m} X_1^{i_1} \cdots X_m^{i_m}$, then

the X_m^d-coeff. is $\sum_{\substack{i_1, \dots, i_m: \\ i_1 + \dots + i_m = d}} r_{i_1 \cdots i_m} c_1^{i_1} \cdots c_{m-1}^{i_{m-1}}$ one of these is $\neq 0$ because $\deg(f) = d$

(the hom. degree d part of f at $(c_1, \dots, c_{m-1}, 1)$).

$\Rightarrow$ By the Nichtnullstellensatz, there are values $c_1, \dots, c_{m-1} \in K$ such that the X_m^d-coeff. $\in K$ is $\neq 0$.

Dividing by this coeff. gives us a monic pol. eq. for X_m in $\Gamma(V)$ with coeff. in $\pi_1^*(\Gamma(K^{m-1}))$.

$\Rightarrow X_m$ is int. over $\pi_1^*(\Gamma(K^{m-1}))$.

$\Rightarrow X_i$ — " —

$X_i - c_i X_m \in \pi_1^*(\Gamma(\dots))$ $\Rightarrow \Gamma(V)$ is int. over $\pi_1^*(\Gamma(K^{m-1}))$. $\square$

Rmk In the Nichtnullstellensatz, we used that K is infinite. There is a generalisation to finite fields that instead of a projection uses a map of the form

$$(a_1, \dots, a_m) \mapsto (a_1 - a_m^{d_1}, \dots, a_{m-1} - a_m^{d_{m-1}})$$

for suitable $d_1, \dots, d_{m-1} \geq 0$.

integers

Cor 11.8 $\dim(V\times W) = \dim(V) + \dim(W)$

Pf Decompose $V = V_1 \cup \dots \cup V_a$,
$W = W_1 \cup \dots \cup W_b$.

Then, $V \times W = \bigcup_{i,j} \underbrace{(V_i \times W_j)}_{\text{irred.}}$.

$\Rightarrow \dim(V\times W) = \max_{i,j}(\dim(V_i\times W_j))$

and $\max_i(\dim(V_i)) + \max_j(\dim(W_j))$
$= \dim(V) + \dim(W)$.

$\Rightarrow$ We can assume that V, W are irreducible.

Let $n = \dim(V)$, $m = \dim(W)$.

$\Rightarrow$ There are surj. fin. morphisms

$\pi: V \to K^n$, $\rho: W \to K^m$.

Claim: $\sigma: V\times W \to K^n \times K^m = K^{n+m}$ is surj. and finite.
$(v,w) \mapsto (\pi(v), \rho(w))$

Pf surj.: clear
fin.: $\Gamma(V) = K[A_1,\dots,A_k]/I$ int. ext. of $\pi^*(K[X_1,\dots,X_n])$
$\Gamma(W) = K[B_1,\dots,B_l]/J$ int. ext. of $\rho^*(K[Y_1,\dots,Y_m])$.
$\Rightarrow \Gamma(V\times W) = K[A_1,\dots,A_k,B_1,\dots,B_l]/(I+J)$ int. ext. of
$\sigma^*(K[X_1,\dots,X_n,Y_1,\dots,Y_m])$.

$\Rightarrow \dim(V\times W) = n+m$. $\square$