10.12

Rmk ~~[illegible]~~ There is in fact a dominant morphism $V \subseteq K^n \longrightarrow K^d$ for $d = \dim(V)$.
Actually, there is a dominant projection $V \longrightarrow K^d$ onto a d-dimensional linear subspace H of K^n spanned by coordinate vectors!

Eg $n=2, d=1$: $\Rightarrow$ proj. onto x- or y-axis is dominant

$n=2, d=2 \Rightarrow$ The map $V \longrightarrow K^2$ is dominant
$\Rightarrow \overline{V} = K^2 \Rightarrow V = K^2$ (↑ V closed)

$n=3, d=2 \Rightarrow$ proj. onto xy- or xz- or yz-plane is dominant

Pf W.l.o.g. V is irred.
The field ext. $K(V)$ of K is generated by $X_1, \dots, X_n$. $\Rightarrow$ There is a transcendence basis of the form $X_{i_1}, \dots, X_{i_d}$. Then, the projection

$$\pi : V \longrightarrow K^d, \quad (x_1, \dots, x_n) \longmapsto (x_{i_1}, \dots, x_{i_d})$$

is dominant because

$$\pi^* : K[Y_1, \dots, Y_d] = \Gamma(K^d) \longrightarrow \Gamma(V), \quad Y_j \longmapsto X_{i_j}$$

is injective because $X_{i_1}, \dots, X_{i_d}$ are algebraically independent over K. $\square$

11. Finite Morphisms

Thm 11.1. Let $f \in K[x_1, \dots, x_n]$

and consider the projection $\pi : V(f) \longrightarrow K^{n-1}$

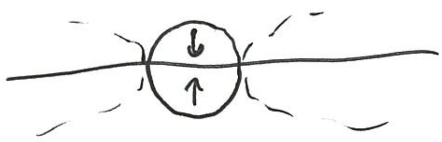
$$(a_1, \dots, a_n) \longmapsto (a_1, \dots, a_{n-1})$$

If f is monic of degree $d \geq 1$ as a pol. in x_n with coeff. in $K[x_1, \dots, x_{n-1}]$, then each $P \in K^{n-1}$ has at least one and at most d preimages $Q \in V(f)$ under the projection map.

Ex: $\pi: V(\underbrace{x^2 + y^2 - 1}_{\text{monic of deg. 2 in } y}) \longrightarrow K$

$$(x, y) \longmapsto x$$

$$\pi^{-1}(t) = \underbrace{\{(t, \pm\sqrt{1-t^2})\}}_{1 \text{ or } 2 \text{ points}}$$



Exe B $\pi : \mathcal{V}(XY-1) \longrightarrow K$

(not monic in Y)

$(x,y) \longmapsto x$

$\pi^{-1}(t) = \{(t, \frac{1}{t})\}$ for $t \neq 0$

$\pi^{-1}(0) = \emptyset$

Exe C $\pi : \mathcal{V}(XY) \longrightarrow K$

(not monic in X)

$(x,y) \longmapsto x$

$\pi^{-1}(t) = \{(t,0)\}$ for $t \neq 0$

$\pi^{-1}(0) = \{(0,y) \mid y \in K\}$

Pf of ~~...~~ Thm 11.1 Write $f(x_1, \dots, x_n) = \sum_{i=0}^{d} f_i(x_1, \dots, x_{n-1}) X^i$

with $f_d = 1$.

Let $(a_1, \dots, a_{d-1}) \in K^{n-1}$.

Then, $(a_1, \dots, a_d) \in \mathcal{V}(f)$ if and only if a_d is a root of the monic pol. $f(a_1, \dots, a_{n-1}, X) = \sum_{i=0}^{d} f_i(a_1, \dots, a_{n-1}) X^i$ of degree d. Such a pol. has ≥ 1 and $\leq d$ roots. $\square$

__Def__ A morphism $\varphi : V \to W$ is __finite__ if the ring ext. $\Gamma(V)$ of $\varphi^*(\Gamma(W))$ is module-finite. ($\varphi^* : \Gamma(W) \to \Gamma(V)$)

__Ex A__ $\pi^* : k[T] \longrightarrow k[X,Y]/(X^2+Y^2-1)$

$T \mapsto X$

$k[X,Y]/(X^2+Y^2-1)$ is mod-fin. over (the subring) $k[X]$ because it is gen. by Y as a ring ext. and Y is int. over $k[X]$.

More generally:

__Prmk__ For f, π as in Prmk 11.1, π is finite.

__Pf__ $k[X_1,\dots,X_n]/(f)$ is gen. by X_n, which is int. over the subring $k[X_1,\dots,X_{n-1}]$ because $\sum_{i=0}^{d} f_i(X_1,\dots,X_{n-1})X_n^i$ is a mon. eq. satisfied by X_n. $\square$

__Ex B, C__ π is not finite.

Ex D $\varphi : K^2 \longrightarrow K^2$ is not finite:

$(x, y) \longmapsto (x, xy)$

Y is not integral over $K[x, xY]$.

[illegible]

[illegible]

Ex If $V \subseteq W$, then the inclusion morphism $V \hookrightarrow W$ is finite (because $\Gamma(W) \to \Gamma(V)$ is surjective).

Rmk The composition of two fin. morphisms is finite.

Pf This follows from the transitivity of module-finiteness. □

Rmk The restriction of a finite morphism $V \to W$ to V' is finite. (It's the composition $V' \hookrightarrow V \to W$.)

Rmk $\varphi : V \to W$ ($W \subseteq K^m$) is finite if and only if

$\varphi : V \to K^m$ is finite.

Pf $\varphi^*(\Gamma(W)) = \varphi^*(\Gamma(K^m))$. □

Thm 11.2 If $\varphi: V \to W$ is a dominant finite morphism, then $\dim(V) = \dim(W)$.

Pf As in the pf of Lemma 10.9, w.l.o.g. V, W are irreducible.

dominant $\Rightarrow \varphi^*: \Gamma(W) \to \Gamma(V)$ injective, we have a hom.

$$\varphi^*: K(W) \hookrightarrow K(V).$$

finite $\Rightarrow$ $\Gamma(V)$ int. ext. of $\Gamma(W)$ (or rather $\varphi^*(\Gamma(W))$)

$\Rightarrow$ $K(V)$ alg. ext. of $K(W)$ (or rather $\varphi^*(K(W))$)

$$\Rightarrow \underbrace{\operatorname{trdeg}(K(V)|K)}_{\dim(V)} = \underbrace{\operatorname{trdeg}(K(W)|K)}_{\dim(W)}$$

□

Cor 11.3 Let $\varphi: V \to W$ be a finite morphism. Then, any point $Q \in W$ has only finitely many preimages $P \in V$.

Pf Assume it has at least one.

$\Rightarrow \varphi|_{\varphi^{-1}(Q)}: \underbrace{\varphi^{-1}(Q)}_{\{P \in V : \varphi(P) = Q\}} \to \{Q\}$ is a finite surjective morphism ($\Rightarrow$ dominant)

$$\Rightarrow \dim(\varphi^{-1}(Q)) = \dim(\{Q\}) = 0.$$

□

[Another proof, we showed for the special case of Thm 11.1:]

Thm 11.4 (lying over property)

Any dominant finite morphism $\varphi: V \to W$ is surjective.

Pf Let $Q \in W$ and let $m := I_W(\{Q\})$ be the corr. maximal ideal of $\Gamma(W)$.

Then, $\varphi^{-1}(Q) = V_V(\varphi^*(m))$. Goal: $\varphi^{-1}(Q) \neq \emptyset$.

Let $I = (\varphi^*(m))$ be the ideal of $\Gamma(V)$ generated by the el. of $\varphi^*(m)$.

By Hilbert's Nsts, it suffices to show that $I \neq \Gamma(V)$.

Assume $I = \Gamma(V)$.

~~Since $\Gamma(V)$ is a fin~~

φ finite $\Rightarrow$ $\Gamma(V)$ fin. gen. as $\varphi^*(\Gamma(W))$-mod.

Let $b_1, \dots, b_r \in \Gamma(V)$ be generators.

$\Rightarrow$ Every el. of $\Gamma(V)$ is a lin. comb. of $b_1, \dots, b_r$ with coeff. in $\varphi^*(\Gamma(W))$.

Every el. of I is a lin. comb. of el. of $\varphi^*(m)$ with coeff. in $\Gamma(V)$.

$\Rightarrow$ " — of el. of the form

$$\underset{m}{\varphi^*(p)}\, b_i\, \underset{\Gamma(W)}{\varphi^*(q)} = \varphi^*(\underbrace{pq}_{\in m})\, b_i .$$

Recall that $I = \Gamma(V) \ni b_i$.

Write $b_i = \sum_j \varphi^*(p_{ij})\, b_j$ with $p_{ij} \in \varphi^*(m)$.

$\Rightarrow$ $Mv = v$ for $M = (\varphi^*(p_{ij}))_{i,j}$, $v = (b_i)_i$.

$\Rightarrow$ $(\mathrm{Id} - M)\, v = 0$

(Id ↑ id. matrix)

$\Rightarrow$ $\det(\mathrm{Id} - M) = 0$

(↑ as in the pf of ...)

The entries of M lie in $\varphi^*(m)$.

$\Rightarrow$ $\det(\mathrm{Id} - M) - 1 \in \varphi^*(m)$

(↑ expand the det; $\det(\mathrm{Id} - M) = 0$)

$\Rightarrow$ $1 \in \varphi^*(m)$.

$\Rightarrow$ $1 \in m$. ↯

(↑ φ dom. $\Rightarrow \varphi^*$ inj)

□

Cor 11.5 Any finite morphism $\varphi: V \to W$ is closed:

The image $\varphi(A) \subseteq W$ of every closed set $A \subseteq V$ is closed.

Ex The proj. $K^2 \to K$ is not closed because the image of $V(XY-1)$ is $K \setminus \{0\}$, which is not closed.

Pf $\varphi: A \to \overline{\varphi(A)}$ is a dominant finite morphism, hence finite. $\Rightarrow \varphi(A) = \overline{\varphi(A)} \Rightarrow \varphi(A)$ is closed. $\square$