Ex X is a tr. basis of $K(\mathcal{V}(X^3-Y^2))$. $\Rightarrow \operatorname{trdeg} = 1$.

(So is Y.)

~~Pf No pol. $f(X)$ lies in $\mathcal{I}(\mathcal{V}(X^3-Y^2))$, for example~~

Pf No pol. $f(X)$ vanishes at every pt. in $\mathcal{V}(X^3-Y^2)$.

(t^2, t^3)

$\Rightarrow X$ is alg. indep.

$\in K(\mathcal{V}(X^3-Y^2))$

Y is alg. over $K(X) \subset K(\mathcal{V}(X^3-Y^2))$.

$\Rightarrow K(\mathcal{V}(X^3-Y^2))$ is alg. over $K(X)$.

$\uparrow$

gen. by X, Y.

$\square$

Thm 10.6 If L is a fin. gen. field ext. of K and M is a fin. gen. field ext. of L, then

$$\operatorname{trdeg}(M|K) = \operatorname{trdeg}(M|L) + \operatorname{trdeg}(L|K).$$

$$\begin{array}{c} M \\ | \\ L \\ | \\ K \end{array}$$

~~Pf~~

Ex ~~trdeg~~ $L = K(x_1, \dots, x_n)$

$$M = L(Y_1, \dots, Y_m) = K(x_1, \dots, x_n, Y_1, \dots, Y_m).$$

Pf If $a_1, \dots, a_n$ is a tr. basis of $L|K$

and $b_1, \dots, b_m$ ... $M|L$,

then $a_1, \dots, a_n, b_1, \dots, b_m$ ... $M|K$:

alg. indep.: follows from Thm 10.1

M alg. over $K(a_1, \dots, a_n, b_1, \dots, b_m)$:

The alg. closure of $K(a_1, \dots, a_n, b_1, \dots, b_m)$ in M contains L and $b_1, \dots, b_m$, hence contains $L(b_1, \dots, b_m)$, hence contains the alg. cl. of $L(b_1, \dots, b_m)$, which is M.

$\square$

Def The *dimension* of an irred. alg. set $V \subseteq K^n$ is $\dim(V) = \mathrm{trdeg}(K(V)|K)$.

Ex $\dim(K^n) = n$

Cor K^n, K^m are not isomorphic (or even birational) unless $n=m$.

Analogy $\mathbb{R}^n, \mathbb{R}^m$ are not homeomorphic unless $n=m$.
More generally, nonempty open subsets $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ are not homeomorphic unless $n=m$.
(These facts are nontrivial!)

Thm 10.7 The dimension of an irred. alg. subset $V = \mathcal{V}(f) \subsetneq K^n$ def. by a single (irred.) pol. $0 \neq f \in K[X_1, \dots, X_n]$ is $\dim(V) = n-1$.

Pf W.l.o.g. the variable X_n occurs in f.
Then, $x_1, \dots, x_{n-1}$ form a tr. basis of $K(V)$ over K:
No pol. $g(x_1, \dots, x_{n-1})$ lies in $(f) = \mathcal{I}(V)$.
x_n is alg. over $x_1, \dots, x_{n-1}$ in $K(V)$ because it satisfies the nonzero pol. eq. $f(x_1, \dots, x_{n-1}, x_n) = 0$ in $K(V)$.

$\square$

Def The *dimension* of a reducible alg. set V with irred. components $V_1, \dots, V_m$ is

$$\dim(V) = \max(\dim(V_1), \dots, \dim(V_m)).$$

The empty set has dimension

$$\dim(\emptyset) = -\infty.$$

Ex $V = \{(0,y) \mid y \in K\} \cup \{(1,2)\}$

has dimension 1.

Thm 10.8 An alg. subset $\emptyset \neq V \subseteq K^n$ has dimension 0 if and only if $|V| < \infty$.

Pf W.l.o.g. V is irreducible.

"$\Leftarrow$" $|V| < \infty$, V irred. $\Rightarrow \underset{\text{(one-point set)}}{|V| = 1} \Rightarrow \Gamma(V) = K$

$\Rightarrow K(V) = K$, which has trdeg $= 0$.

"$\Rightarrow$" $\dim(V) = 0 \Rightarrow K(V)$ is an alg. ext. of K

$\underset{K \text{ alg. cl.}}{\Rightarrow} K(V) = K \Rightarrow \Gamma(V) = K \Rightarrow I(V)$ max. ideal of $K[X_1, \dots, X_n]$

$\Rightarrow |V| = 1$.

$\square$

Lemma 10.9 If V, W are alg. sets and

a) there is a dominant morphism $\varphi: V \to W$ or

b) V is irred. and there is a dom. rat. map $\varphi: V \dashrightarrow W$,

then $\dim(V) \geq \dim(W)$.

Pf Decompose V, W into irred. comp.:

$V = V_1 \cup \dots \cup V_a$, $\quad W = W_1 \cup \dots \cup W_b$.

$$W = \overline{\varphi(V)} = \bigcup_{i=1}^{a} \overline{\varphi(V_i)}$$

$$\Rightarrow \underbrace{W_j}_{\text{irred.}} = \bigcup_i \underbrace{(\overline{\varphi(V_i)} \cap W_j)}_{\text{closed}} \qquad \forall j$$

$\Rightarrow W_j = \overline{\varphi(V_{r_j})} \cap W_j$ for some r_j.

$\Rightarrow W_j \subseteq \overline{\varphi(V_{r_j})}$

Since $\overline{\varphi(V_{r_j})} \subseteq W$ is irred., it is contained in some W_{s_j}.

$\Rightarrow W_j \subseteq \overline{\varphi(V_{r_j})} \subseteq W_{s_j}$

$\Rightarrow j = s_j$, $W_j = \overline{\varphi(V_{r_j})}$

$\Rightarrow$ We obtain dom. rat. maps $\varphi: V_{r_j} \dashrightarrow W_j$,

so can assume w.l.o.g. that V, W are irred.

We then have a K-alg. hom. $\varphi^*: K(W) \hookrightarrow K(V)$.

If $a_1, \dots, a_d$ is a tr. basis of $K(W)$, then $\varphi^*(a_1), \dots, \varphi^*(a_d) \in K(V)$

are still alg. indep.

$\square$

Lemma 10.10 If $V \subseteq W$, then $\dim(V) \leq \dim(W)$.

Pf W.l.o.g. V is irred. (replace V by an irred. comp.)
W.l.o.g. W is irred. (replace W by an irred. comp. containing V)
The inclusion $i : V \hookrightarrow W$ induces a surjective ring hom.

$$i^* : \Gamma(W) \longrightarrow \Gamma(V)$$
$$f \longmapsto f \circ i = f|_V$$

Since the el. of $\Gamma(V)$ generate the field ext. $K(V)$ of K, by Lemma 10.2, there are elements $a_1, \dots, a_d \in \Gamma(V)$ which form a tr. basis of $K(V)$.
Pick any preimages $b_1, \dots, b_d \in \Gamma(W)$ (ext. to W).
They are ~~…~~ alg.-independent because their restrictions to V are.
$a_1, \dots, a_d$

$\square$

__Thm 10.11__ Let $V \subseteq K^n$ be an irred. alg. set. Then, $\dim(V)$ is the largest number $d \geq 0$ such that there is a dom. rat. map $\varphi : V \dashrightarrow K^d$.

__Pf__ Rat. maps $V \dashrightarrow K^n$ corr. to tuples $(f_1, \ldots, f_d)$ of rat. functions $f_1, \ldots, f_d \in K(V)$.

Such a rat. map is dominant if and only if the hom.
$$K[X_1, \ldots, X_d] \longrightarrow K(V) \text{ is injective.}$$
$$X_i \longmapsto f_i$$

That's equiv. to the alg. independence of $f_1, \ldots, f_d$. $\square$