Rmk If $\varphi: V \to W$ is a dominant morphism, we obtain an injective ring hom.

$$\varphi^*: \Gamma(W) \to \Gamma(V)$$
$$f \mapsto f \circ \varphi$$

which induces a field hom.

$$\varphi^*: K(W) \longrightarrow K(V)$$
$$\frac{a}{b} \longmapsto \frac{\varphi^*(a)}{\varphi^*(b)}$$
$$f \longmapsto f \circ \varphi$$

Rmk Dominance of φ is important:
Otherwise, f might not be defined at any point in $\varphi(V)$, so $f \circ \varphi$ might not be defined anywhere!
(Or: $\varphi^*(b)$ might be 0 for $b \neq 0$.
$\Rightarrow \varphi^*(\frac{1}{b}) = \frac{1}{\varphi^*(b)}$ undefined.)

Rmk $\quad U_{\varphi^*(f)} \supseteq \underbrace{\varphi^{-1}(U_f)}_{\text{open subset of } V} \neq \emptyset$

Def For any open subset $\emptyset \neq U \subseteq V$, let $\mathcal{O}_V(U)$ be the ring (!) of rational functions $f \in K(V)$ defined at every point in U.

For $U = \emptyset$, let $\mathcal{O}_V(U) = 1$, the one-element ring.

Rmk Elements of $\mathcal{O}_V(U)$ correspond to functions $f: U \to K$ which are (locally) given by a quotient of regular functions.

Rmk If $\varphi: V \to W$ is any morphism between alg. sets and $U \subseteq W$ is any open subset, we obtain a ring hom.

$$\varphi^*: \mathcal{O}_W(U) \longrightarrow \mathcal{O}_V(\varphi^{-1}(U)).$$

Def Let $V \subseteq K^n$ and $W \subseteq K^m$ be alg. subsets and let V be irreducible.

A rational map $\varphi: V \dashrightarrow W$ is a map $U \to W$ for an open subset U of V which is given by m rational functions $f_1, \dots, f_m$ defined on U, where we identify two such maps $U_1 \to W$, $U_2 \to W$ if they agree on the (nonempty) intersection $U_1 \cap U_2$.

Ex
$$K^3 \dashrightarrow K^2$$
$$(x,y,z) \mapsto \left(\frac{x}{y+z^2}, \frac{1}{z}\right)$$

Ex
$$\mathcal{V}(X^3 - Y^2) \dashrightarrow K$$
$$(x,y) \mapsto \frac{y}{x}$$

Ex Any morphism $V \to W$.

Ex A rat. map $V \dashrightarrow K^m$ is the same as m rational functions on V.

Def A rat. map $\varphi : V \dashrightarrow W$ is <u>dominant</u> if $\overline{\varphi(U)} = W$ for any/every open $\emptyset \neq U \subseteq V$ where φ is defined.

Rmk If $\varphi : A \dashrightarrow B$ and $\psi : B \dashrightarrow C$ are rat. maps, and φ is dominant, we get a rat. map
$\psi \circ \varphi : A \dashrightarrow C$.

In particular: If $\varphi : V \dashrightarrow W$ is dominant, we get a field hom. $\varphi^* : K(W) \to K(V)$,
$f \mapsto f \circ \varphi$.

Rmk We get a bijection

$$\left\{ \begin{matrix} \text{dominant} \\ \text{rational} \\ \text{map} \\ V \dashrightarrow W \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} K\text{-algebra} \\ \text{hom.} \\ K(W) \to K(V) \end{matrix} \right\}.$$

Pf Same as for morphisms:
$f_i = \varphi^*(x_i)$, where $x_i \in \Gamma(W) \subseteq K(W)$ is the i-th coordinate map. $\square$

Def (irreducible alg. sets) V, W are <u>birational</u> if there are dominant rational maps $\varphi : V \dashrightarrow W$ and $\psi : W \dashrightarrow V$ with $\psi \circ \varphi = id_V$ and $\varphi \circ \psi = id_W$.

Ex K and $\mathcal{V}(x^3 - y^2) \subset K^2$ are not isomorphic (Problem 1d on Pset 4), but are birational:

$$\varphi : K \longrightarrow \mathcal{V}(x^3 - y^2)$$
$$t \mapsto (t^2, t^3)$$

$$\psi : \mathcal{V}(x^3 - y^2) \dashrightarrow K$$
$$(x,y) \mapsto \frac{y}{x} \quad (\text{def. on } \mathcal{V}(x^3 - y^2) \setminus \{(0,0)\})$$

Rmk V, W are birational if and only if the K-algebras $K(V), K(W)$ are isomorphic.

10. Dimension and transcendence degree

Def Let $L|K$ be a field extension.
Elements $a_1, \dots, a_n \in L$ are <u>algebraically dependent</u> over K if there is a pol. $0 \neq f \in K[X_1, \dots, X_n]$ such that $f(a_1, \dots, a_n) = 0$.

Rmk For $n=1$, this is equiv. to a_1 being algebraic over K.

Ex $X_1, \dots, X_n \in K(X_1, \dots, X_n)$ are alg. independent over K.

Ex $X, Y \in K(V(X^3 - Y^2))$ are alg. dep. because $X^3 - Y^2 = 0$ in $K(V(X^3 - Y^2))$.

Ex π, e are transcendental over $\mathbb{Q}$.
It is unknown whether π, e are alg. indep. over $\mathbb{Q}$.

Thm 10.1 $a_1, \dots, a_n \in L$ are alg. dep. if and only if some a_i is alg. over $K(a_1, \dots, a_{i-1})$.

Analogy Let V be a K-vector space.
$v_1, \dots, v_n \in V$ are linearly dep. if and only if some v_i is a lin. comb. of $v_1, \dots, v_{i-1}$.

Pf "$\Leftarrow$" ~~...~~ a_i alg. over $K(a_1, \dots, a_{i-1})$

$\Rightarrow f(a_i) = 0$ for some $0 \neq f \in K(a_1, \dots, a_{i-1})[X]$.

Each coeff. of f is the quotient of two pol. in $a_1, \dots, a_{i-1}$ with coeff. in K.

We can clear out denominators so the coeff. are pol. in $a_1, \dots, a_{i-1}$.

"$\Rightarrow$" Let ~~...~~ $f(a_1, \dots, a_n) = 0$ for some $0 \neq f \in K[X_1, \dots, X_n]$.

Let $g(X_n) = f(a_1, \dots, a_{n-1}, X_n)$. $(\Rightarrow g(a_n) = 0)$

<u>Case 1:</u> $g \in K[X_n]$ is not the zero pol.

$\Rightarrow a_n$ is alg. over K

<u>Case 2:</u> $g \in K[X_n]$ is the zero pol.

Write $f = \sum_i f_i(X_1, \dots, X_{n-1}) \cdot X_n^i$. Some $f_m \neq 0$.

$\Rightarrow \underset{0=}{g(X_n)} = \sum_i f(a_1, \dots, a_{n-1}) \cdot X_n^i$. $\Rightarrow f_m(a_1, \dots, a_{n-1}) = 0$.

$\Rightarrow a_1, \dots, a_{n-1}$ are alg. dep.

Proceed by induction over n. $\square$

Def Elements $a_1, \dots, a_n \in L$ form a <u>transcendence basis</u> of L over K if they are alg. indep. and L is an alg. ext. of $K(a_1, \dots, a_n)$.

Rmk A transcendence basis is a maximal list of alg. indep. el. of K (a list to which we can't append any el. of L without destroying algebraic independence).

Ex $X_1, \dots, X_n$ is a tr. basis of $K(X_1, \dots, X_n)$ over K.

Lemma 10.2 ("Exchange lemma")

If $a_1, \dots, a_n \in L$ are alg. indep. and L is alg. over $K(b_1, \dots, b_m)$ (with $b_1, \dots, b_m \in L$), then there are indices $1 \le i_1 < \dots < i_r \le m$ $(r \geq 0)$ such that $a_1, \dots, a_n, b_{i_1}, \dots, b_{i_r}$ form a transcendence basis of L over K.

Pf Choose any max. alg. indep. sublist of $a_1, \dots, a_n, b_1, \dots, b_m$ containing $a_1, \dots, a_n$. The remaining b_j are alg. over K (this list) $=: F$.

$\Rightarrow K(a_1, \dots, a_n, b_1, \dots, b_m)$ is alg. over F

Also, L is alg. over $K(b_1, \dots, b_m) \subseteq K(a_1, \dots, a_n, b_1, \dots, b_m)$.

$\Rightarrow L$ is alg. over F. $\Rightarrow$ The list forms a tr. basis. $\square$

__Cor 10.3__ Any finitely generated field ext. has a (finite) tr. basis.

__Thm 10.4__ If $a_1,\dots,a_n$ is a tr. basis of L over K and $b_1,\dots,b_m \in L$ are alg. indep., then $n \geq m$.

__Analogy__ If $v_1,\dots,v_n$ span V and $w_1,\dots,w_m \in V$ are lin. indep., then $n \geq m$.

__Cor 10.5__ Any two tr. bases of L over K have the same size, called the __transcendence degree__ of L over K.

trdeg$(L|K)$

__Rmk__ $\operatorname{trdeg}(L|K) = 0 \Leftrightarrow L$ is an alg. ext. of K.

__Ex__ $\operatorname{trdeg}(K(X_1,\dots,X_n)|K) = n$.

__Pf of Thm 10.4__ Use ind. over n.

__$n=0$:__ $\Rightarrow L$ is alg. over K

$\Rightarrow$ There are no alg. indep. el.

$\Rightarrow m=0$.

__$n-1 \to n$:__ ~~Use~~ By the exchange lemma, b_1 together with some of the a_i form a tr. basis of L over K.

W.l.o.g. $b_1, a_1, \dots, a_r$ do.

We have $r<n$ because $a_1, \dots, a_n$ already form a tr. basis, so b_1 is alg. over $K(a_1, \dots, a_n)$.

Now, $a_1, \dots, a_r$ form a tr. basis of L over $K(b_1)$.

Also, $b_2, \dots, b_m \in L$ are alg. indep. over $K(b_1)$.

The ind. hypothesis shows that $\underset{\substack{\wedge\!\shortparallel \\ n-1}}{r} \geq m-1$.

$\Rightarrow n \geq m$. $\square$