Buchberger's Algorithm

We can compute a finite Gröbner basis of $I = (f_1, \dots, f_m)$ as follows:

Construct sets

$$F = G_0, \quad G_1, \quad G_2, \dots$$

of polynomials generating I such that

$$(\mathrm{lm}(G_0)) \subsetneq (\mathrm{lm}(G_1)) \subsetneq (\mathrm{lm}(G_2)) \subsetneq \cdots$$

If G_k fails Buchbergers Criterion, there is a <u>reduction</u> $r \neq 0$ of some $S(g_1, g_2)$ with $g_1, g_2 \in G_k$ (w.r.t. G_k).

$\Rightarrow$ $\mathrm{lm}(r)$ is not divisible by any element of $\mathrm{lm}(G_k)$.

Take $G_{k+1} = G_k \cup \{r\}$.

$\Rightarrow$ $(\mathrm{lm}(G_{k+1})) \supsetneq (\mathrm{lm}(G_k))$.

By Hilbert's Basis Theorem, this process terminates after a finite number of steps. $\blacksquare$

<u>Rmk</u> You can also after each step replace any elements g of G_k by its reduction w.r.t. $G_k \setminus \{g\}$, one polynomial g at a time.

Ex $I = (XY^2, X^2Y+1)$, lex. order

$f_1 = XY^2 \qquad G_0 = \{f_1, f_2\}$

$f_2 = X^2Y+1$

$r = S(f_1,f_2) = X\cdot f_1 - Y\cdot f_2 = -Y$

is reduced w.r.t. $\{f_1, f_2\}$.

$G_1 = \{\cancel{f_1}, f_2, r\}$

$f_2 + X^2\cdot r = 1$

$G_1' = \{1\}$.

is a gröbner basis

Ex $I = (X^3-2XY, X^2Y-2Y^2+X)$, deg. lex. order

$f_1 = X^3 - 2XY$

$f_2 = X^2Y - 2Y^2 + X$

$r = S(f_1,f_2) = Y\cdot f_1 - X\cdot f_2 = -2XY^2 + 2XY^2 - X^2$

$= -X^2$

is reduced w.r.t. $\{f_1, f_2\}$

$G_1 = \{f_1, f_2, r\}$

$f_1' = f_1 + X\cdot r = -2XY$

$G_1' = \{f_1', f_2, r\}$

$f_2' = f_2 + Y\cdot r = -2Y^2 + X$

$G_1'' = \{f_1', f_2', r\}$

$$S(f_1', f_2') = Y \cdot f_1' - X \cdot f_2' = -X^2$$

reduces to 0 w.r.t. $\{f_1', f_2', r\}$

$$S(f_1', r) = X \cdot f_1' - 2Y \cdot r = 0$$

$$S(f_2', r) = X^2 \cdot f_2' - 2Y^2 \cdot r = X^3$$

reduces to 0 w.r.t. $\{f_1', f_2', r\}$

$\Rightarrow$ $\{f_1', f_2', r\}$ is a Gröbner basis.
$= \{-2XY, -2Y^2+X, -X^2\}$

In particular, $I \neq (1)$, so $V(I) \neq \emptyset$.

Often, deg. rev. lex. order is faster than lex. order.

9. Rational functions

Let $V \subseteq K^n$ be an irreducible alg. set.

Recall that $\Gamma(V)$ is an integral domain.

<u>Def</u> The <u>field of rational functions</u> on V is the field of fractions $K(V)$ of $\Gamma(V)$. Formally:

the set of equivalence classes of pairs (a,b) with $a,b \in \Gamma(V)$, $b \neq 0$, where $(a,b) \sim (a',b')$ if $ab' = a'b$. (A pair (a,b) corr. to the fraction $\frac{a}{b}$.)

$K(V) = \{ \frac{a}{b} \mid a,b \in \Gamma(V), b \neq 0 \}$

not everywhere 0 on V

<u>Ex</u> $V = K^n \rightsquigarrow K(V) = K(X_1, \dots, X_n)$

<u>Ex A</u> $V = \mathcal{V}(XY - Z^2) \subset K^3$

Note that $\frac{X}{Z} = \frac{Z}{Y}$ in $K(V)$ because $XY = Z^2$ in $\Gamma(V)$.

<u>Def</u> A rational function $f \in K(V)$ is <u>defined at</u> $P \in V$ if $f = \frac{a}{b}$ for some $a,b \in \Gamma(V)$ with $b(P) \neq 0$.

In this case, we write $f(P) = \frac{a(P)}{b(P)} \in K$.

(Note: The value $f(P)$ doesn't depend on the choice of a,b.)

<u>Ex A</u> $f = \frac{X}{Z} = \frac{Z}{Y}$ is defined at all points $(x,y,z) \in V$ with $z \neq 0$ <u>or</u> $y \neq 0$.

Rmk If $f = \frac{a}{b}$ for some $a, b \in \Gamma(V)$ with $a(P) \neq 0$ and $b(P) = 0$, then f is not defined at P.

Pf Assume $\frac{a}{b} = \frac{a'}{b'}$ with $b'(P) \neq 0$. Then,

$$\underbrace{a(P)}_{\neq 0}\underbrace{b'(P)}_{\neq 0} = a'(P)\underbrace{b(P)}_{=0}. \qquad \lightning \qquad \square$$

Ex A $f = \frac{x}{z} = \frac{z}{y}$ is not defined at $(x,0,0) \in V$ for any $x \neq 0$.

Lemma 9.1 The set of points $P \in V$ at which $f \in K_V$ is not defined is closed.
Equivalently, the set $U_f \subseteq V$ of points at which f is defined is open (w.r.t. the subspace top. on V).
Also, $U_f \neq \emptyset$.

Pf Write $f = \frac{a}{b}$. Since $b \neq 0$ as a fct. on V, it is not zero everywhere on V. $\Rightarrow U_f \neq \emptyset$.

$$\{P \in V \mid f \text{ not def. at } P\} = \bigcap_{\substack{a,b \in \Gamma(V): \\ f = \frac{a}{b}}} \underbrace{V_V(b)}_{\text{closed}} \quad \text{is closed.}$$

$\square$

Ex A f is not defined at $(0,0,0)$, which lies in the closure of $\{(x,0,0) \mid x \neq 0\}$.

For $K = \mathbb{C}$, for example, the limit of $f(P)$ as $P \to (0,0,0)$ can depend on the path!

$$\begin{tikzcd} P \doteq (t,t,t) \in V \arrow[r, squiggly] \arrow[d, "t \to 0"] & f(P) = \frac{t}{t} = 1 \arrow[d] \\ (0,0,0) & 1 \end{tikzcd}$$

$$\begin{tikzcd} P \doteq (t^{3/2}, t^{1/2}, t) \in V \arrow[r, squiggly] \arrow[d, "t \to 0"] & f(P) = \frac{t^{3/2}}{t} = t^{1/2} \arrow[d] \\ (0,0,0) & 0 \end{tikzcd}$$

$\Rightarrow$ no cont. ext. to $(0,0,0)$

__Summary:__ Any $f \in K(V)$ gives rise to a map $f : U_f \to K$.

__Lemma 9.2__ If $f \in K(V)$ is zero on a nonempty open subset $U \subseteq U_f$, then $f = 0$.

__Pf__ Write $f = \frac{a}{b}$. For any $P \in U$, we have $b(P) = 0$ or $a(P) = 0$.

$$\Rightarrow V = \underset{V^*}{(V \setminus U)} \cup \underset{V^*}{\mathcal{V}_V(b)} \cup \mathcal{V}_V(a)$$

closed

Since V is irreducible, this implies $\mathcal{V}_V(a) = V$, so [redacted] $a = 0$. $\square$

__Cor 9.3__ If $f, g \in K(V)$ agree on a nonempty open subset $U \subseteq U_f \cap U_g$, then $f = g$.

__Rmk__ This is similar to a fact from complex analysis: If two meromorphic functions [redacted] on a connected region U agree on a nonempty open subset $U' \subseteq U$, they agree on U.

__Cor 9.4__ The elements of $K(V)$ correspond bijectively to equiv. classes of pairs (U, f) with $\emptyset \neq U \subseteq V$ open and $f : U \to K$ any map which is locally given by a quotient of regular functions

(i.e.: [redacted] $\forall P \in U \ \exists P \in U' \subseteq U$ open, $a, b \in \Gamma(V)$:
$$\forall Q \in U' : b(Q) \neq 0, \ f(Q) = \frac{a(Q)}{b(Q)}.),$$

where we identify $(U, f), (U', f')$ if $f|_{U \cap U'} = f'|_{U \cap U'}$.
(Recall that the intersection of two nonempty open subsets of an irred. set is nonempty!)