Cor 6.4 a) If $V \subseteq K^n$ is finite, then $\dim_{K\text{-vector space}}(\Gamma(V)) = |V|$.

b) If $V \subseteq K^n$ is infinite, then — " — $= \infty$.

Pf a) $\Gamma(V) = \{f : V \to K\} \cong K^{|V|}$

b) If $P_1, \dots, P_m$ are distinct points in V, then we get a surjection

$$\begin{array}{ccc} \Gamma(V) & \twoheadrightarrow & \Gamma(\{P_1, \dots, P_m\}) \\ f & \longmapsto & f|_{\{P_1, \dots, P_m\}} \end{array}$$

m-dimensional.

$\square$

7. Morphisms

Def Let $V \subseteq K^n$ and $W \subseteq K^m$ be algebraic subsets.

A morphism (= regular map = polynomial map) $\varphi : V \to W$ is a map $V \to W$ which is given by polynomials: There exist $f_1, \dots, f_m \in K[X_1, \dots, X_n]$ such that $\varphi(P) = (f_1(P), \dots, f_m(P)) \in W$ $\forall P \in V$.

Ex $f : K \longrightarrow V(X^2 - Y) \subseteq K^2$
$x \longmapsto (x, x^2)$

Ex A projection $K^n \longrightarrow K$.
$(a_1, \dots, a_n) \mapsto a_i$

Ex The identity $id : V \to V$.

Ex An inclusion $V \hookrightarrow W$, where $V \subseteq W$.

Rmk If $\varphi : A \to B$ and $\psi : B \to C$ are morphisms, then the composition $\psi \circ \varphi : A \to C$ is a morphism.

Rmk Morphisms $\varphi : \underset{\subseteq K^n}{V} \longrightarrow K^m$ correspond exactly to tuples $(f_1, \dots, f_m)$ of functions $f_i \in \Gamma(V)$.
$(f_i : V_i \to K)$

In particular, morphisms $\varphi : V \to K$ correspond exactly to functions $f \in \Gamma(V)$.

How to tell whether the image of $\varphi : V \overset{\subseteq K^n}{\longrightarrow} K^m$ is contained in W? $\underbrace{}_{\text{corr. to } f_1,\dots,f_n}$

<u>Lemma 7.1</u> $\varphi(V) \subseteq W$ if and only if

$$h(f_1(x_1,\dots,x_n),\dots,f_m(x_1,\dots,x_n)) \in I(V)$$

for all $h \in I(W)$ if φ corr. to pol. $f_1,\dots,f_m \in K[x_1,\dots,x_n]$.

<u>Pf</u> $\varphi(P) \in W = V(I(W)) \quad \forall P \in V$

$$\Leftrightarrow \underbrace{h(\varphi(P))}_{= h(f_1(P),\dots,f_m(P))} = 0 \quad \forall h \in I(W),\ P \in V$$

$$\Leftrightarrow h(f_1,\dots,f_m) \in I(V) \quad \forall h \in I(W).$$

$\square$

<u>Def</u> For any morphism $\varphi : V \to W$, its <u>pullback function</u> is the K-algebra homomorphism

$$\varphi^* : \Gamma(W) \longrightarrow \Gamma(V).$$
$$f \longmapsto f \circ \varphi$$

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[dr, dashed, "f\circ\varphi = \varphi^*(f)"'] & W \arrow[d, "f"] \\ & K \end{tikzcd}$$

(Sometimes, φ^* is denoted by $\tilde{\varphi}$ instead.)

[redacted]

Other description: If $\varphi = (f_1, \dots, f_m)$

$$\varphi^* : K[Y_1, \dots, Y_m]/I(W) \longrightarrow K[X_1, \dots, X_n]/I(V)$$

$$[g(Y_1, \dots, Y_m)] \longmapsto [\, g(f_1(X_1, \dots, X_n), \dots, f_m(X_1, \dots, X_n))\,]$$

<u>Thm 7.2</u> We get a bijection

$$(\text{morphisms } V \to W) \longleftrightarrow (K\text{-alg. hom. } \Gamma(W) \to \Gamma(V)).$$

$$\varphi \longmapsto \varphi^*$$

<u>Pf</u> How to determine φ from φ^*?

Let $Y_i \in \Gamma(W)$ be the function mapping any point in $W \subseteq K^m$ to its i-th coordinate.

Then, $\varphi(P) = (Y_1(\varphi(P)), \dots, Y_m(\varphi(P)))$

$$= (\underbrace{\varphi^*(Y_1)}(P), \dots, \underbrace{\varphi^*(Y_m)}(P))$$

The m elements of $\Gamma(V)$ defining the morphism φ. □

<u>Ex</u> $\varphi : K \longrightarrow V(X^2 - Y)$, $t \longmapsto (t, t^2)$

$\varphi^* : \Gamma(V(X^2 - Y)) \longrightarrow \Gamma(K) = K[T]$

$X \longmapsto T$

$Y \longmapsto T^2$

Rmk a) $\mathrm{id}: V \to V$ corresponds to $\mathrm{id}: \Gamma(V) \to \Gamma(V)$.

b) $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[dr, dashed, "\varphi^* \circ \psi^*(f)"'] & B \arrow[r, "\psi"] \arrow[d, dashed, "\psi^*(f)"] & C \arrow[dl, "f"] \\
& K &
\end{tikzcd}$$

Rmk Hence,

$$\left\{\begin{array}{l}\text{alg. subsets of } K^n \\ \text{for some } n\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{fin. gen.} \\ \text{reduced} \\ K\text{-algebra}\end{array}\right\}$$

$$V \longmapsto \Gamma(V) = K[x_1, \dots, x_n]/J(V)$$

$$\varphi \longmapsto \varphi^*$$

is a contravariant equivalence of categories.

Def A morphism $\varphi: V \to W$ is an <u>isomorphism</u> if it has an inverse morphism $\psi: W \to V$ (with $\varphi \circ \psi = \mathrm{id}_W$, $\psi \circ \varphi = \mathrm{id}_V$).

Rmk $\varphi: V \to W$ is an isomorphism if and only if $\varphi^*: \Gamma(W) \to \Gamma(V)$ is an isomorphism.

Ex The inverse of
$$K \longrightarrow V(x^2 - y)$$
$$x \longmapsto (x, x^2)$$
is
$$x \longleftarrow (x, y).$$

Ex Any translation in K^n is an isomorphism.

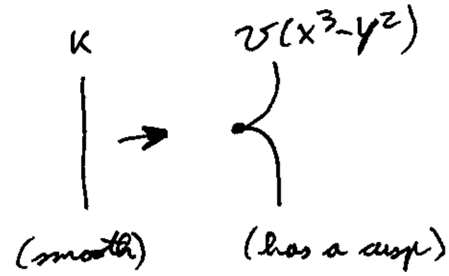
Ex Any invertible linear map $K^n \to K^n$ is an isomorphism.

__Warning__ Not every bijective morphism is an isomorphism!

(Just like not every bijective continuous map is a homeomorphism.)

__Ex__ $\varphi : K \longrightarrow \mathcal{V}(x^3-y^2) \subseteq K^2$

$t \longmapsto (t^2, t^3)$



is a bijection

~~Because~~ $\cancel{\tfrac{y}{x} \mapsto (x,y) \text{ iff } x^3-y^2=0 \text{ and } (x,y)\neq 0}$

~~and~~ $\cancel{0 \mapsto (0,0)}$

with inverse map

$$\frac{y}{x} \longleftarrow (x,y) \neq (0,0),$$

$$0 \longleftarrow (0,0).$$

It is not an isomorphism because

$$\varphi^* : \Gamma(\mathcal{V}(x^3-y^2)) \longrightarrow \Gamma(K) = K[X]$$

$$X \longmapsto T^2$$

$$Y \longmapsto T^3$$

isn't a isomorphism because T is not contained in the image.

Thm 7.3 Any morphism $\varphi : V \subseteq K^n \longrightarrow W \subseteq K^m$ is continuous (w.r.t. the Zariski topologies).

Pf For any ideal $J \subseteq \Gamma(W)$,

$$\varphi^{-1}(\underbrace{\mathcal{V}_W(J)}_{\substack{\text{closed}\\ \text{subset}\\ \text{of } W}}) = \{P \in V \mid \underbrace{\varphi(P) \in \mathcal{V}_W(J)}\} = \underbrace{\mathcal{V}_V(\varphi^*(J))}_{\substack{\text{closed}\\ \text{subset}\\ \text{of } V}}$$

$$\Updownarrow$$

$$f(\varphi(P)) = 0 \ \forall f \in J$$

$$\Updownarrow$$

$$\varphi^*(f)(P) = 0 \ \forall f \in J$$

$$\Updownarrow$$

$$P \in \mathcal{V}_V(\varphi^*(J))$$

$\square$

Thm 7.4 $\varphi^* : \Gamma(W) \longrightarrow \Gamma(V)$ is injective if and only if $\varphi(V)$ is (Zariski) dense in W.

Def Such a morphism is called dominant.

Pf Let $V \subseteq K^n$ and $W \subseteq K^m$.

φ^* injective

$$\Leftrightarrow \forall f \in \Gamma(W): \text{ if } \underset{f\circ\varphi}{\varphi^*(f)} = \underset{(\text{on } V)}{0}, \text{ then } f = \underset{(\text{on } W)}{0}.$$

$$\Updownarrow$$

$$f = 0 \text{ on } \varphi(V)$$

$$\Leftrightarrow \mathcal{I}_W(\varphi(V)) = 0$$

$$\| $$

$$\mathcal{I}_W(\overline{\varphi(V)})$$

$$\Leftrightarrow \overline{\varphi(V)} = W.$$

bijection $\{\text{alg. subset of } W\} \longleftrightarrow \{\text{radical ideal of } \Gamma(W)\}$

$\square$