Lemma 2.2

a) For any collection of ideals I_α, (of $K[x_1,\dots,x_n]$)

$$\bigcap_\alpha V(I_\alpha) = V\Big(\bigcup_\alpha I_\alpha\Big)$$

$$= V\Big(\text{ideal gen. by } \bigcup_\alpha I_\alpha\Big)$$

b) For any two ideals I, J [redacted],

$$V(I) \cup V(J) = V(I \cap J) = V(\underbrace{I \cdot J}_{})$$

ideal generated by polynomials of the form $f \cdot g$ with $f \in I$, $g \in J$

c) $V(0) = K^n$

d) $V(1) = \emptyset$

Pf a) clear

b) $V(I)\cup V(J)=V(I\cdot J)$:

$p\in LHS \Leftrightarrow p\in V(I)$ or $p\in V(J)$

$\Leftrightarrow \forall f\in I: f(p)=0$ or $\forall g\in J: g(p)=0$

$\Leftrightarrow \forall f\in I, g\in J: f(p)=0$ or $g(p)=0$

$\Leftrightarrow p\in RHS$

$V(I)\cup V(J)\subseteq V(I\cap J)$:

clear

$V(I\cap J)\subseteq V(I\cdot J)$:

clear because $I\cdot J\subseteq I\cap J$.

c) $p\in V(0)\Leftrightarrow 0=0$

d) $p\in V(1)\Leftrightarrow 1=0$ $\square$

Cor 2.3

a) [redacted] The intersection of arbitrarily many alg. subsets of K^n is alg.

b) The union of two (or finitely many) alg. subsets of K^n is alg.

c) K^n is an alg. subset

d) $\emptyset$ is an alg. subset

Hence, the alg. subsets of K^n are the closed [redacted] sets of a topology on K^n, which is called the Zariski topology.

~~ ~~

<u>Ex</u> Every finite subset of K^n is Zariski closed because every one-point subset is.

<u>Lemma 2.4</u> If $K = \mathbb{R}$ or $\mathbb{C}$ and $X \subseteq K^n$ is Zariski closed, then $X \subseteq K^n$ is closed w.r.t. the usual (Euclidean) topology on K^n.

<u>Pf</u> For any $f \in K[x_1, \dots, x_n]$, the set $V(f) = f^{-1}(\{0\})$ of zeros of f is closed w.r.t. the usual topology because $f: K^n \to K$ is continuous w.r.t. the usual topology and $\{0\} \subseteq K$ is closed.

$$\Rightarrow V(I) = \bigcap_{f \in I} \underbrace{V(f)}_{\text{closed}} \quad \text{is closed for any } I.$$

$\square$

Thm 2.5 The alg. subsets of K (for $n=1$) are:

K and the finite subsets of K

In other words, the Zariski topology on K is the cofinite topology.

Pf Consider any ideal I of $K[X]$.

The ring $K[X]$ is a principal ideal domain (in fact a unique factorization domain) because you can perform the Euclidean algorithm in $K[X]$.

$\Rightarrow I=(f)$ for some $f \in K[X]$.

Case 1: $f=0$ (constant zero polynomial)

$\Rightarrow V(I)=V(0)=K$

Case 2: $f=0$

$\Rightarrow f$ has only finitely many roots. $\square$

Ex of Lemma 2.4

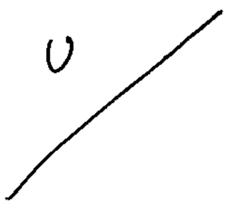
$K=\mathbb{R}$, $n=1$

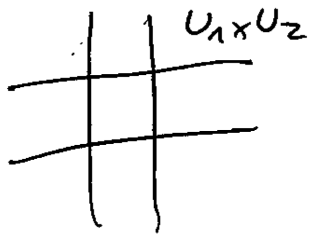
zar. closed: $\underbrace{\mathbb{R}, \text{fin. subsets}}_{\text{closed w.r.t. usual top.}}$

Warning The Zariski topology on K^n is (in general) not the product topology arising from the product topology on K.

~~[illegible]~~

Ex $U = \mathbb{C}^2 \setminus \{(x,y) \in \mathbb{C}^2 \mid x = y\}$ is Zariski open, but doesn't contain any (nonempty) product $U_1 \times U_2$ with $\underbrace{U_1}_{= \mathbb{C} \setminus \text{fin. many pts}}, \underbrace{U_2}_{= \mathbb{C} \setminus \text{fin. many pts.}} \subseteq \mathbb{C}$ Zariski open.





3. Hilbert's Basis Theorem

Goal: Every alg. set is defined by finitely many polynomial equations.

Convention: Rings are commutative and have a multiplicative unit 1.

Def A ring R is noetherian if every ideal I of R is generated by finitely many elements.

Ex Any principal ideal domain is noetherian.
(e.g. any field K or pol. ring over a field $K[x]$)

Lemma 3.1 R is noetherian if and only if there is no chain of ideals

$$I_1 \subsetneqq I_2 \subsetneqq I_3 \subsetneqq \cdots$$

Pf "$\Rightarrow$" $I := \bigcup_{r \geq 1} I_r$ is an ideal of R.

Let $I = (f_1, \ldots, f_m)$.
Each f_i lies in some I_r.
$\Rightarrow I \subseteq I_r$ for some r.
$\Rightarrow I_r = I_{r+1} = \cdots$

"$\Leftarrow$" Assume I isn't finitely generated.

$\Rightarrow$ We can inductively construct

$0 \subsetneq (f_1) \subsetneq (f_1, f_2) \subsetneq (f_1, f_2, f_3) \subsetneq \dots$

by taking any $f_r \in I \setminus (f_1, \dots, f_{r-1})$. $\square$

<u>Thm 3.2</u> (Hilbert's Basis Theorem)

If R is noetherian, then $R[X]$ is noetherian.

By induction, this implies:

<u>Cor 3.3</u>

If R is noetherian, then $R[X_1, \dots, X_n]$ is noetherian.

<u>Cor 3.4</u>

Any alg. subset $V \subseteq K^n$ is defined by finitely many polynomial equations: $V = \mathcal{V}(f_1, \dots, f_r)$.

Pf of Thm 3.2

Assume $I \subseteq R[X]$ isn't finitely generated.

We inductively construct

$$0 \subsetneq (f_1) \subsetneq (f_1, f_2) \subsetneq \dots$$

by taking $f_r \in I \setminus (f_1, \dots, f_{r-1})$

of minimum degree.

Let $d_r = \deg(f_r)$.

$\Rightarrow d_1 \leq d_2 \leq \dots$

Let $b_r = lc(f_r)$.

The leading coefficient $lc(h)$ of a (nonzero) pol. $h(x) = a_n x^n + \dots + a_0$ with $a_n \neq 0$ is a_n.

We get a chain of ideals of R:

$$0 \subseteq (b_1) \subseteq (b_1, b_2) \subseteq \dots$$

Since R is noetherian, we have equality somewhere:

$$(b_1, \dots, b_r) = (b_1, \dots, b_r, b_{r+1}).$$

$\Rightarrow b_{r+1} \in (b_1, \dots, b_r)$

$\rightsquigarrow$ Write $b_{r+1} = b_1 c_1 + \dots + b_r c_r$ with $c_1, \dots, c_r \in R$.

$$\Rightarrow g(X) := \underbrace{f_{r+1}(X)}_{\substack{\deg = d_{r+1} \\ lc = b_{r+1} \\ \in I \\ \notin (f_1, \dots, f_r)}} - \sum_{i=1}^{r} \underbrace{f_i(X) \cdot c_i \cdot X^{d_{r+1} - d_i}}_{\substack{\deg = d_{r+1} \\ lc = b_i c_i \\ \in I \\ \in (f_1, \dots, f_r)}}$$

has degree $\deg(g) < d_{r+1} = \deg(f_{r+1})$.

But $g \in I \setminus (f_1, \dots, f_r)$, contradicting the assumption that f_{r+1} has minimum degree among the elements of $I \setminus (f_1, \dots, f_r)$. $\square$

__Warning__ For every $n \geq 1$, there are ideals of $k[x,y]$ that aren't generated by n elements!