Algebraic Geometry

1. Overview

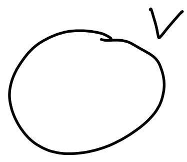
Let K be a field.

An <u>algebraic subset</u> of K^n is the set of solutions $(x_1,\dots,x_n)\in K^n$ to a system of polynomial equations:

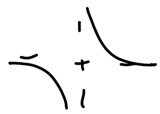
$$f_1(x_1,\dots,x_n)=0, \quad f_1\in K[x_1,\dots,x_n]$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$f_m(x_1,\dots,x_n)=0, \quad f_m\in K[x_1,\dots,x_n]$$

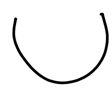
<u>Ex</u>

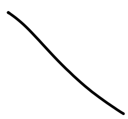
Conic (= conic section):

Circle $V=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\}$ 

Ellipse $2x^2+3y^2=1$ 

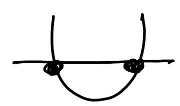
Hyperbola $xy=1$ 

Parabola $y=x^2$ 

Line $x+2y=3$ 

Point $\{(1,2)\}=\{(x,y)\in\mathbb{R}^2 \mid y=2x,\ x+1=y\}$ 

$=\{\qquad \mid x=1,\ y=2\}$

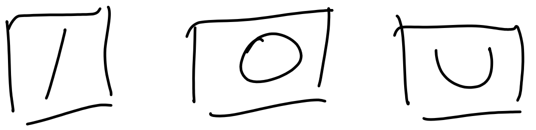
Two points $\{(0,0),(1,0)\}=\{\qquad \mid x(x-1)=y,\ y=0\}$ 

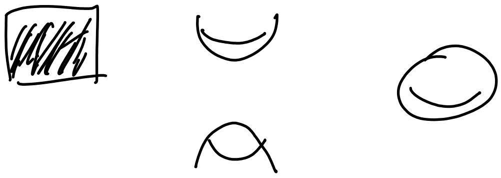
$\vdots$

Questions

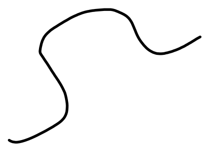
- Is V a set of just finitely many points?
  If so, how many?
- What is the "dimension" of V?

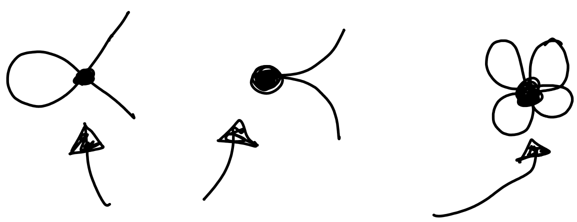
dim $= 0$: 

dim $= 1$: 

dim $= 2$: 

- Is V "smooth"?

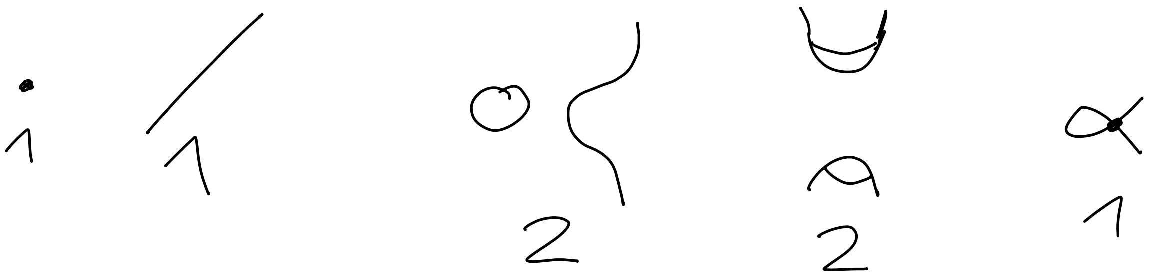
smooth 

not smooth 

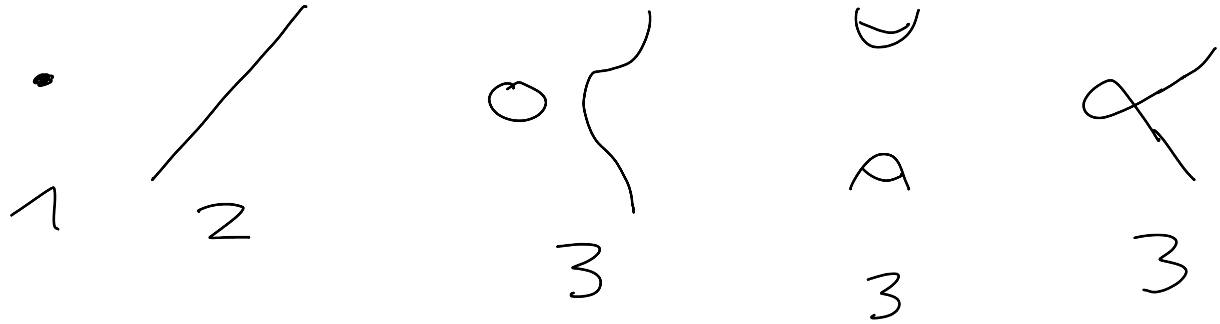
- If not, what are the "singularities" look like?

Real algebraic geometry ($K = \mathbb{R}$)

- How many connected components does V have?

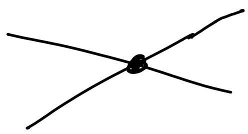


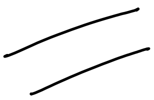
- How many connected components does the complement $\mathbb{R}^n \setminus V$ have?



Intersection theory

- In how many points do two lines $L_1 \neq L_2 \subseteq K^2$ intersect?

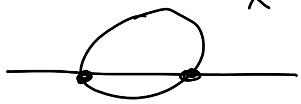
Usually 1 

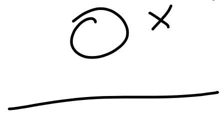
Occasionally 0  (if L_1, L_2 are parallel)

$\rightsquigarrow$ Always 1 in the projective plane.

- In how many points does a line intersect a conic?

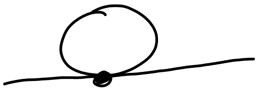
$x^2+(y-2)^2=100$

Sometimes 2  $y=1$

Sometimes 0  $x^2+(y-2)^2=100$

$y=-1000$

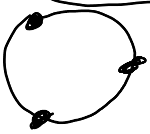
(can't happen in algebraically closed fields like $\mathbb{C}$)

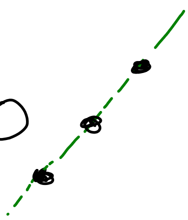
Occasionally 1  (with "multiplicity" 2)

Enumerative geometry

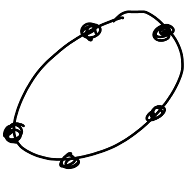
- How many circles are there through three given points p_1, p_2, p_3?

(distinct)

Usually 1 

Occasionally 0 

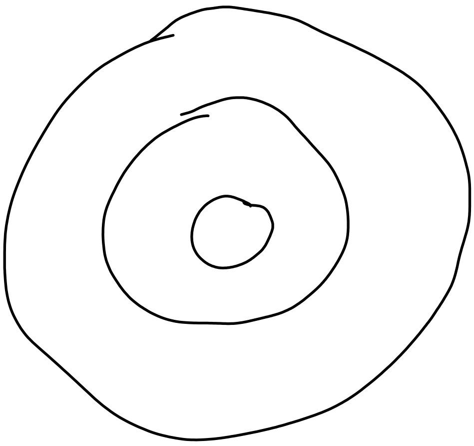
- How many conics are there through five given points p_1, p_2, p_3, p_4, p_5?

Usually 1 

- How many circles are there tangent to three given circles?

Sometimes 8

Sometimes 0



1

2

3

4

5

6

But 7 is impossible!

- How many lines are there that intersect four given lines in three-dimensional space?

  Usually 2

- How many lines are there on a given cubic surface (surface defined by a pol. of degree 3)?

  Usually 27 (if $K = \mathbb{C}$).

Prerequisites

Algebra: rings, modules, fields, ...
algebraically closed fields

References

Fulton
Brooke Ullery's lecture notes

Grade

70% weekly homework
(dropping the two lowest scores)

30% take-home exam

OH this week: Mo, Th 4-5pm in 233

Course assistants:

Raphael Tsiamis (rtsiamis@college.harvard.edu)
Raluca Vlad (rvlad@college.harvard.edu)

2. Basic definitions

Let K be a field.

<u>Rmk</u> In algebraic geometry, the set of points in K^n is often also denoted by $\mathbb{A}^n$ or $\mathbb{A}^n_K$ and called the n-dimensional <u>affine space</u> (over K).

<u>Def</u> The <u>vanishing locus</u> of a set $S \subseteq K[x_1, \dots, x_n]$ of polynomials is the corresponding set of zeros:

$$\mathcal{V}(S) = \{ P \in K^n \mid f(P) = 0 \ \forall f \in S \}$$

<u>Ex</u> $(n=2)$ $\mathcal{V}(\{x_2 - x_1^2\}) = \{(x_1, x_2) \in K^2 \mid x_2 = x_1^2\}$

<u>Ex</u> $\mathcal{V}(\{x_1 - a_1, \dots, x_n - a_n\}) = \{(a_1, \dots, a_n)\}$

<u>Def</u> The <u>vanishing ideal</u> of a set $X \subseteq K^n$ is the set of polynomials that vanish everywhere on X:

$$\mathcal{I}(X) = \{ f \in K[x_1, \dots, x_n] \mid f(P) = 0 \ \forall P \in X \}$$

<u>Lemma 2.1</u> $\mathcal{I}(X)$ is an ideal of $K[x_1, \dots, x_n]$.

<u>Pf</u>
- If $\underset{(\Leftrightarrow f \in \mathcal{I}(X))}{f(P) = 0}$, then $\underset{(\Leftrightarrow fg \in \mathcal{I}(X))}{f(P) \cdot g(P) = 0}$ for all $g \in K[x_1, \dots, x_n]$.
- If $\underset{(\Leftrightarrow f \in \mathcal{I}(X))}{f(P) = 0}$ and $\underset{(\Leftrightarrow g \in \mathcal{I}(X))}{g(P) = 0}$, then $\underset{(\Leftrightarrow f+g \in \mathcal{I}(X))}{f(P) + g(P) = 0}$.

$\square$

<u>Ex</u> $(n=2)$

$$\mathcal{I}(\{(0,0)\}) = \{ f(x_1, x_2) \in K[x_1, x_2] \mid \underset{\substack{\uparrow \\ \text{const. coeff. of } f}}{f(0,0)} = 0 \} = \underset{\substack{\uparrow \\ \text{ideal generated by } f}}{(x_1, x_2)}$$

__Rmk__ $\mathcal{V}$ is inclusion-reversing:

if $S \subseteq T$, then $\mathcal{V}(S) \supseteq \mathcal{V}(T)$

$\mathcal{I}$ is inclusion-reversing:

if $X \subseteq Y$, then $\mathcal{I}(X) \supseteq \mathcal{I}(Y)$

__Rmk__ If I is the ideal generated by $S \subseteq k[x_1, \dots, x_n]$,

then $\mathcal{V}(I) = \mathcal{V}(S)$.

__Pf__ "$\subseteq$" follows from $I \supseteq S$

"$\supseteq$" Every element of I can be written as

$$\underbrace{\underbrace{f_1}_{0 \text{ at } P} g_1 + \dots + \underbrace{f_r}_{0 \text{ at } P} g_r}_{0 \text{ at } P}$$

with $f_1, \dots, f_r \in S$ and $g_1, \dots, g_r \in k[x_1, \dots, x_n]$

for all $P \in S$. $\square$

Def A subset $X \subseteq K^n$ is algebraic if $X = \mathcal{V}(S)$ for some $S \subseteq K[x_1, \dots, x_n]$.

Rmk This differs from the definition in chapter 1, where we only allowed finitely many polynomial equations. We'll soon see that the two definitions are equivalent!

Ex $\{(x_1, x_2) \mid x_2 = x_1^2\}$

Ex Any one-point subset $\{P\} \subseteq K^n$.