Math 223b: Algebraic Number Theory

Spring 2021

Problem set #8

due Friday, April 9 at noon

We assume that K is a number field throughout this problem set. Furthermore, E, E_1, E_2 are elliptic curves defined over K.

Problem 1. Let V and W be complete varieties. Show that $V \times W$ is complete.

Problem 2. Let $\phi : E_1 \to E_2$ be a nonzero isogeny defined over K with $\ker(\phi) \subseteq E_1(K)$. Let K' be the field of definition of a point $P \in E_1(\overline{K})$ with $Q = \phi(P) \in E_2(K)$. Show that $K'|K$ is an Galois extension whose Galois group is isomorphic to a quotient group of $\ker(\phi)$.

Problem 3. Let $n \geqslant 1$. Show that there is a number $A_n \geqslant 1$ such that every nonarchimedean local field k of characteristic zero whose residue field characteristic does not divide n has at most A_n field extensions of degree n (up to isomorphism).

Problem 4. Consider the morphism $\varphi : \mathbb{G}_m \to \mathbb{G}_m$ sending $x \in \overline{K}^\times \cong \mathbb{G}_m(\overline{K})$ to $x^3 \in \overline{K}^\times \cong \mathbb{G}_m(\overline{K})$. Let K' be the field of definition of $x \in \mathbb{G}_m(\overline{K})$ with $\phi(x) \in \mathbb{G}_m(\mathcal{O}_K)$. Show that $K'|K$ is unramified at all primes not dividing 3.

Problem 5. Let $a, b \in K$ with $b \neq 0$ and $a^2 - 4b \neq 0$. We then obtain elliptic curves

$$E_1 = \{[x : y : z] \in \mathbb{P}^2_K \mid y^2 z = x(x^2 + axz + bz^2)\}$$

and

$$E_2 = \{[x : y : z] \in \mathbb{P}^2_K \mid y^2 z = x(x^2 - 2axz + (a^2 - 4b)z^2)\}$$

and an isogeny $\phi : E_1 \to E_2$ sending $[x : y : z]$ to $[y^2 z : y(x^2 - bz^2) : x^2 z]$ for $[x : y : z] \neq [0 : 1 : 0], [0 : 0 : 1]$.

a) Show that the kernel of ϕ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \cong \{\pm 1\}$. What are the points in the kernel?

b) Consider the Kummer theory isomorphism

$$\mathbb{Q}^\times/\mathbb{Q}^{\times 2} \cong \mathrm{Hom}(\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}), \{\pm 1\})$$

sending x to $\sigma \mapsto \sigma(\sqrt{x})/\sqrt{x}$. Consider its composition with the injective morphism

$$E_2(K)/\phi(E_1(K)) \hookrightarrow \mathrm{Hom}(\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}), \{\pm 1\})$$

defined in the proof of the weak Mordell-Weil theorem. Show that the image of a point $[x : y : 1] \in E_2(K)$ with $x \neq 0$ in $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$ is the coset containing x. (Also, note that clearly $x \in \mathbb{Q}^{\times 2}$ if $[x : y : 1] \in \phi(E_1(K))$!)

c) Assume that $a, b \in \mathbb{Z}$ and let x as in b). Show that the coset $x\mathbb{Q}^{\times 2}$ in $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$ contains an integer dividing $a^2 - 4b$.

Problem 6 (Infinite bonus, Jacobian conjecture)**.** Let $\varphi : K^2 \to K^2$ be a morphism whose derivative $D\varphi(P) : K^2 \to K^2$ at every point $P \in K^2$ is invertible ("unramified at every point"). Does this imply that φ is an isomorphism?