Math 223b: Algebraic Number Theory

Spring 2021

Problem set #7

due Monday, March 29 at noon

We assume that K is a number field throughout this problem set. Furthermore, E, E_1, E_2 are elliptic curves defined over K.

Problem 1. Let G be an abelian group and let $f : G \to \mathbb{R}$ be a function satisfying the parallelogram law

$$f(x+y) + f(x-y) = 2(f(x) + f(y))$$

for all $x, y \in G$. Show that the map $\langle \cdot, \cdot \rangle : G \times G \to \mathbb{R}$ given by $\langle x, y \rangle = \frac{1}{2}(f(x+y) - f(x) - f(y))$ is bilinear.

Problem 2 (Kronecker's theorem)**.** Let $P = [x : y] \in \mathbb{P}^1(\overline{\mathbb{Q}})$. Show that $h(P) = 0$ if and only if $[x^n : y^n] = [1 : 1]$ for some $n \geqslant 1$.

Problem 3 (Northcott's theorem)**.** Let C be a smooth projective curve (defined over K) and let $\varphi : C \to C$ be a nonconstant morphism (defined over K) of degree at least 2. We denote its n-th iterate $\varphi \circ \cdots \circ \varphi$ by φ^n. Show that there are only finitely many points $P \in C(K)$ such that for some $n > m \geqslant 1$, we have $\varphi^n(P) = \varphi^m(P)$.

Problem 4. Let $\phi : E_1 \to E_2$ be a nonzero isogeny (defined over K). Show that $\widehat{h}(\phi(P)) = \deg(\phi)\widehat{h}(P)$ for all $P \in E_1(\overline{K})$.

Problem 5. a) Show that the group $E(\overline{K})$ is not finitely generated.

b) (bonus) Show that the $\mathbb{Q}$-vector space $E(\overline{K}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is infinite-dimensional. (Hint: Use the Chebotarev density theorem to find points $P_1, \ldots, P_n$ of the form $P_i = (x_i : y_i : 1)$ with $x_i \in K$ and $[K(y_1, \ldots, y_n) : K] = 2^n$. Show that they are linearly independent in $E(\overline{K}) \otimes_{\mathbb{Z}} \mathbb{Q}$.)