Math 223b: Algebraic Number Theory

Spring 2021

Problem set #4

due Friday, March 5 at noon

We assume that the field K has characteristic 0 throughout this problem set.

Problem 1. Let $C \subseteq \mathbb{P}^2_K$ be a smooth projective curve of degree d (defined by a homogeneous polynomial in $K[X,Y,Z]$ of degree d). Let $l \subseteq \mathbb{P}^2_K$ be a line defined by a linear equation $r(X,Y,Z) = 0$, which intersects C in d distinct points $P_1, \ldots, P_d$. Let $D = P_1 + \cdots + P_d$.

a) Show that for $k \geqslant 0$, the set $L(kD)$ consists exactly of the functions of the form $f = \frac{a}{r^k}$ for a homogeneous polynomial $a(X,Y,Z)$ of degree k.

b) Show that we have

$$l(kD) = \begin{cases} 0, & k < 0, \\ \frac{1}{2}(k+2)(k+1), & 0 \leqslant k < d, \\ \frac{1}{2}d(2k-d+3), & k \geqslant d. \end{cases}$$

c) Show that $g_C = \frac{1}{2}(d-1)(d-2)$.

Problem 2. Let C be a smooth projective curve of genus 0 defined over K and assume that $C(L) \neq \varnothing$ for some field extension L of K of odd degree. Show that $C(K) \neq \varnothing$.

Problem 3. Let C be a smooth curve of genus 1 defined over K and let $D \in \mathrm{Div}(C)$ be a divisor of degree 4. We obtain a closed embedding $\varphi : C \to \mathbb{P}^3_K$ associated to a basis of $L(D)$. Show that $\varphi(C)$ is the intersection $A \cap B$ of two hypersurfaces $A, B \subset \mathbb{P}^3_K$ of degree 2.

Problem 4. Find a projective (irreducible) curve C over some field K and a rational function $f \in K(C)$ such that there is no morphism $\varphi : C \to \mathbb{P}^1_K$ with $\varphi(P) = [f(P) : 1]$ for all points $P \in C(\overline{K})$ at which f is defined.