Math 223b: Algebraic Number Theory

Spring 2021

Problem set #3

due Friday, February 26 at noon

We assume that the field K has characteristic 0 throughout this problem set.

Problem 1. Consider the smooth projective curve $C = V(X^3+Z^3-Y^2Z) \subset \mathbb{P}^2_{\mathbb{Q}}$ and the morphism $f : C \to \mathbb{P}^1_{\mathbb{Q}}$ given by $[x : y : z] \mapsto [x : z]$ for $(x,z) \neq (0,0)$. Compute the degree of f and its ramification divisor R_f.

Problem 2. Let $C = V(f) \subset \mathbb{P}^2_K$ be a smooth curve defined by an irreducible homogeneous polynomial $f(X,Y,Z) \in K[X,Y,Z]$ of degree d. Show that there is a divisor of degree d on C defined over K.

Problem 3. Give an example of a smooth projective curve C defined over $\mathbb{Q}$ which has no divisor of degree 1 defined over $\mathbb{Q}$.

Problem 4. Let A, B, C be smooth projective curves and let $f : A \to B$ and $g : B \to C$ be nonconstant morphisms. Show that $R_{g\circ f} = R_f + f^*(R_g)$.

Problem 5. Write $\mathbb{P}^1_{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. We call a morphism $f : \mathbb{P}^1_{\mathbb{Q}} \to \mathbb{P}^1_{\mathbb{Q}}$ a *Belyi map* for a subset $S \subseteq \mathbb{P}^1_{\mathbb{Q}}$ if $f(S) \subseteq \{0, 1, \infty\}$ and the divisor $f(R_f)$ is of the form $a[0] + b[1] + c[\infty]$ with $a, b, c \in \mathbb{Z}$.

a) Show that there is a Belyi map for $S = \{0, 1, \infty, t\}$ for every $t \in \mathbb{Q}$.

b) Show that there is a Belyi map for every finite set $S \subseteq \mathbb{P}^1_{\mathbb{Q}}(\mathbb{Q})$.

c) (bonus) Show that there is a Belyi map for every finite set $S \subseteq \mathbb{P}^1_{\mathbb{Q}}(\overline{\mathbb{Q}})$.