Math 223b: Algebraic Number Theory

Spring 2021

Problem set #2

due Friday, February 19 at noon

We assume that the field K has characteristic 0 throughout this problem set.

Problem 1. Let $f(X) \in K[X]$ be a nonconstant polynomial. Consider the affine variety $V = V(f(X) - Y^2) \subseteq \mathbb{A}^2_K$.

a) Show that V is irreducible if and only if $f(X)$ is not the square of a polynomial.

b) Show that V is smooth if and only if the polynomial $f(X)$ is squarefree.

Problem 2. Let $V \subseteq \mathbb{A}^n_K$ be an irreducible variety (defined over K) and let $S \subseteq V$ be the set of singular points of V.

a) Show that S is an affine variety defined over K.

b) (bonus) Show that $S \subsetneq V$.

Problem 3. Let I be an ideal of $K[X_1, \ldots, X_n]$ and $A = K[X_1, \ldots, X_n]/I$. Let F be the free A-module with basis $(\mathrm{d}X_1, \ldots, \mathrm{d}X_n)$ and let $Q = \mathrm{d}I$ be the set of elements $\mathrm{d}f$ of F with $f \in I$, where we let

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial X_i} \cdot \mathrm{d}X_i.$$

a) Show that Q is an A-module.

b) Show that the module of differentials $\Omega_K(A)$ is isomorphic to F/Q (where the isomorphism sends the element $\mathrm{d}X_i$ of $\Omega_K(A)$ defined in class to the element $\mathrm{d}X_i$ of F/Q).

c) Assuming that the ideal I is generated by $f_1, \ldots, f_m$, show that Q is generated by $\mathrm{d}f_1, \ldots, \mathrm{d}f_m$.

Problem 4. Consider the smooth curve $V = V(X^3 + 17 - Y^2) \subseteq \mathbb{A}^2_K$. Feel free to use a computer for the following problems, but try to do them without using Bézout's theorem.

a) Compute the roots $P \in V$ and their multiplicities $v_{V,P}(f_1)$ of the function $f_1(X,Y) = X - 3Y + 13$ on V.

b) Compute the roots $P \in V$ and their multiplicities $v_{V,P}(f_2)$ of the function $f_2(X,Y) = 3X - 8Y + 35$ on V.

Problem 5 (bonus)**.** Let $V \subseteq \mathbb{A}^n_K$ be a smooth (irreducible) curve and let $P \in V(K)$. Show the following claims from class:

a) The ring $\mathcal{O}_{V,P}$ is a discrete valuation ring with residue field K.

b) Let $v_{V,P}$ be the normalized discrete valuation. An element f of $\mathcal{O}_{V,P}$ has $v_{V,P}(f) \geqslant 1$ if and only if the value $f(P)$ is zero and has $v_{V,P}(f) \geqslant 2$ if and only if the derivative $Df(P) : T_{V,P} \cong K \to K$ is zero.