Math 223b: Algebraic Number Theory

Spring 2021

Problem set #1

due Monday, February 8 at noon

Problem 1. Consider a matrix $M \in \mathrm{GL}_{n+1}(\mathbb{Q})$. It induces an automorphism φ of the n-dimensional projective space $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q}^{n+1}/\mathbb{Q}^\times$. Show that $H(\varphi(P)) \asymp_M H(P)$ for all $P \in \mathbb{P}^n(\mathbb{Q})$.

(In other words: Show that there exists a constant $C = C(M) > 0$ such that $H(\varphi(P)) \geqslant C \cdot H(P)$ and $H(P) \geqslant C \cdot H(\varphi(P))$.)

Problem 2. Let $N(T)$ be the number of pairs $(x, y) \in \mathbb{Z}^2$ with $x^2 - 3y^2 = 1$ and $|x|, |y| \leqslant T$. Show that $N(T) \asymp \log T$ for $T \to \infty$.

Problem 3 (Dirichlet's approximation theorem)**.** Prove that for any $\alpha \in \mathbb{R}$ and $N \geqslant 1$, there exists some rational number $\frac{p}{q} \in \mathbb{Q}$ with $|q| \leqslant N$ and $|p - q\alpha| < \frac{1}{N}$.

Problem 4 (Liouville's theorem)**.** Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be algebraic with minimal polynomial $f(X) \in \mathbb{Q}[X]$ of degree n. Show (without using Roth's theorem) that there exists $C > 0$ such that for all $\frac{p}{q} \in \mathbb{Q}$ with $p, q \in \mathbb{Z}$, we have

$$|p - q\alpha| \geqslant \frac{C}{|q|^{n-1}}.$$

Hint: Find an upper bound for the rational number $|f(p/q)|$.

Problem 5. Let $f(X, Y) \in \mathbb{Q}[X, Y]$ be a squarefree homogeneous polynomial of degree $n \geqslant 2$. Assume that $f(X, Y)$ has a root in $\mathbb{P}^1(\mathbb{C})\backslash\mathbb{P}^1(\mathbb{R})$. Assuming Roth's theorem, show that for any $r \in \mathbb{Q}^\times$, the equation $f(x, y) = r$ has only finitely many solutions $(x, y) \in \mathbb{Z}^2$.