Rmk: The constructed bijection

$$V^G(\overline{K}) \longleftrightarrow (G\text{-orbit in } V(\overline{K}))$$

is $Gal(\overline{K}|K)$-equivariant.

In particular, it restricts to a bijection

$$V^G(K) \longleftrightarrow (G\text{-orbits } S \text{ in } V(\overline{K}) \text{ with } \sigma(S)=S \ \forall \sigma \in Gal(\overline{K}|K))$$

3.5. The Jacobian variety (cont.)

Reference: Lang, Abelian Varieties, II.2

Let C be a smooth projective curve over K of genus g and fix a point $P \in C(K)$.

Recall the surjective and "almost injective" map

$$d: C^{(g)}(\overline{K}) \longrightarrow Cl^0(C_{\overline{K}})$$

$$[(Q_1,\dots,Q_g)] \longmapsto [Q_1]+\dots+[Q_g]-g[P]$$

(which is a bijection if C is an ell. curve)

and the corr. map

$$D: C^{(g)}(\overline{K}) \longrightarrow Div^0(C_{\overline{K}})$$

$$[(Q_1,\dots,Q_g)] \longmapsto [Q_1]+\dots+[Q_g]-g[P],$$

Lemma 3.5.1 There are rational maps

$$\alpha : C^{(g)} \times C^{(g)} \dashrightarrow C^{(g)}$$
$$\beta : C^{(g)} \dashrightarrow C^{(g)}$$

such that $d(\alpha(x,y)) = d(x) + d(y)$ for (x,y) in a dense open subset of $C^{(g)} \times C^{(g)}$

and $d(\beta(x)) = -d(x)$ for x in a dense open subset of $C^{(g)}$.

Rmk This is not even obvious when d is a bijection. (For ell. curves C, we explicitly constructed the group op. α, β on C.)

Trick Let V be an irred. affine variety over K. Let $L = K(V)$ be its field of rational functions. Denote by V_L the variety V over the base field $L \supseteq K$. Then, $V_L(L)$ contains a "natural" point T called the generic point:

Consider an embedding $V \subseteq \mathbb{A}^n_K$, so

$V = \{Q \in \mathbb{A}^n_K \mid f(Q) = 0 \ \forall f \in I\}$ for some set $I \subseteq K[x_1, \dots, x_n]$

$V_L = \{ Q \in \mathbb{A}^n_L \mid f(Q) = 0 \ \forall f \in I \}$.

Let a_i be the image of X_i in the field of fractions L of $K(X_1, \dots, X_n)/I$.

Let $T = (a_1, \dots, a_n) \in L^n$.

Note that $f(X_1, \dots, X_n) \equiv 0 \mod I$, so

$$f(a_1, \dots, a_n) = 0 \text{ in } L$$
for all $f \in I$.

$\Rightarrow T \in V_L(L)$.

This single point $T \in V_L(L)$ with coordinates in L encodes information about all points $Q \in V(K)$ with coord. in K because can "<u>specialize</u>" T to Q by plugging the coordinates of Q in for the variables that are the coordinates of T.

<u>Pf of Lemma</u> Let U be an affine patch of C.

$U \subseteq C$ dense open

$U^{(g)} \subseteq C^{(g)}$ dense open

Let $L = K(C^{(g)}) = K(U^{(g)})$

$$= (K(\underbrace{U \times \cdots \times U}_{g}))^{S_g} = (K(\underbrace{C \times \cdots \times C}_{g}))^{S_g}.$$

Let $T \in U^{(g)}_L(L)$ be the generic point.

We obtain a "generic divisor"

$$D(T) \in \mathrm{Div}^0(C_L).$$

Consider the divisor $E = -D(T) + g[P] \in \mathrm{Div}(C_L)$ of degree g.

Apply R-R to this divisor (over the base field L!).

$\Rightarrow \ell(E) \geq 1.$

the vector space, not the field L!

Let $f_1, \ldots, f_r \in K(C_L)$ be a basis of $\mathcal{L}(E)$.

There is a dense open subset $U' \subset U^{(g)}$ such that for all $S \in U'$, plugging the coordinates of S into the coeff. of $f_1, \ldots, f_r$ (which are elements of L and therefore rational functions on C) and into the coordinates of the points in the divisor E (which are also elements of L) produces well-defined elements $f_1, \dots, f_r \in K(C)$ and $E \in \mathrm{Div}(C)$ and $f_1^{(S)}, \dots, f_r^{(S)}$ are linearly independent (because linear dependence is a polynomial condition, which doesn't hold for all $S \in C^{(g)}$ because $f_1, \dots, f_r \in K(C_L)$ were linearly independent) and

$$f_1^{(S)}, \dots, f_r^{(S)} \in L(\underbrace{E^{(S)}}_{-D(S)+g[P]}).$$

<u>Claim</u> There is a dense open subset $U'' \subset U^{(g)}$ such that for all $S \in U''$, we have $\ell(\underbrace{-d(S) + g[P]}_{\deg = g}) = 1$.

<u>Pf</u> In the proof of "almost-injectivity" (in the first part of 3.4), we saw that there is a dense open subset $U''' \subset \underbrace{U \times \dots \times U}_{g}$ such that

$$\ell(2g[P] - Q_1 - \dots - Q_g) = 1 \text{ for all } (Q_1, \dots, Q_g) \in U'''.$$

Let U'' be the image of U''' in $U^{(g)}$. $\square$

Since $U', U'' \subset U^{(g)}$ are dense open subsets, $U' \cap U'' \neq \emptyset$.

On U', $\ell \geq r$. On U'', $\ell = 1$.

$\Rightarrow$ $r = 1$, so $\ell(E) = 1$.

Write $f = f_1$ for the generator of $L(E)$.

Then, $\underbrace{E + \mathrm{div}(f)}_{\text{div. of degree } g} \geq 0$.

$\rightsquigarrow$ Write $E + \mathrm{div}(f) = Q_1 + \dots + Q_g$

with $Q_1, \dots, Q_g \in C_L(\bar{L})$

!!!

Since $Q_1 + \dots + Q_g \in \mathrm{Div}(C_L)$ is $\mathrm{Gal}(\bar{L}|L)$-invariant, the multiset $\{Q_1, \dots, Q_g\}$ is $\mathrm{Gal}(\bar{L}|L)$-invariant, so the tuple $(Q_1, \dots, Q_g)$ corr. to a point $T' \in C_L^{(g)}(L)$. $\leftarrow$ (important!)

$$\underset{\substack{\parallel \\ Q_1 + \dots + Q_g}}{E + \mathrm{div}(f)} = -D(T) + g[P] + \mathrm{div}(f).$$

$\Rightarrow$ $-D(T) + \mathrm{div}(f) = D(T')$, so

$-D(T)$ and $D(T')$ lie in the same divisor class on C_L.

The coord. of $T' \in C_L(L)$ are elements of L and therefore rational functions on $C^{(g)}$. They define a rational map

$\beta : C^{(g)} \dashrightarrow C^{(g)}$.

There is a dense open subset $U''' \subset U^{(g)}$ such that for all $S \in U'''$, plugging the coord. of S into the coord. of T' and into the coeff. of f produces well-def. $\beta(S) = T'^S \in C^{(g)}$ and $f^{(S)} \in K(C)$ with

$$-D(S) + \mathrm{div}\left(f^{(S)}\right) = D(\underbrace{T'^S}_{\beta(S)}),$$

so $-D(S)$ lies in the same divisor class as $D(\beta(S))$.

$\rightsquigarrow$ $d(\beta(S)) = -d(S)$ for $S \in U'''$.

For constructing α, use the same technique, over the field

$$L = K(C^{(g)} \times C^{(g)}).$$

Only difference:

<u>Claim</u> There is a dense open subset $U'' \subset U^{(g)} \times U^{(g)}$ such that for all $(S_1, S_2) \in U''$, we have

$$\ell(\underbrace{d(S_1) + d(S_2) + g[P]}_{\deg.\ g}) = 1.$$

<u>Pf</u> Let W be a canonical divisor.

R-R: $\ell(D) - \ell(W-D) = \deg(D) + 1 - g$.

for all divisors $D \in \mathrm{Div}(C)$.

Also, $\ell(\underbrace{W + g[P]}_{\deg = 3g-2}) = 2g-1$.

As in the proof of "almost-injectivity", there is a dense open subset

$$U'' \subset \underbrace{U \times \dots \times U}_{g} \times \underbrace{U \times \dots \times U}_{g} \text{ such that}$$

$$\ell(W + g(P) - (Q_1) - \dots - (Q_{2g-1})) = 0$$

for all $(Q_1, \dots, Q_g) \in U''$

$$\parallel$$

$$\ell(W + g(P) - (Q_1) - \dots - (Q_{2g}))$$

$$\underset{R-R}{\Longrightarrow} \ell(((Q_1) + \dots + (Q_g) - g(P)) + ((Q_{g+1}) + \dots + (Q_{2g}) - g(P))$$

$$+ g(P))$$

$$= \ell(Q_1 + \dots + Q_{2g} - g(P))$$

$$= g + 1 - g + 0 = 1$$

Let U' be the image of U'' in $U^{(g)} \times U^{(g)}$.

$\square$

" $\square$ "