C. Quotients of varieties by finite groups

Reference: Joe Harris, Alg. Geom. (A first course), lecture 10

Q Let G be a finite group acting on a variety V (def. over K) such that for any $g \in G$, the map $\tau_g : V \longrightarrow V$, $x \mapsto gx$ is a morphism (def. over K).

Assume $V \subseteq \mathbb{A}^n_K$.
The morphism $\tau_g : V \to V$ corr. to a K-alg. hom. $\tau_g^* : \Gamma(V) \to \Gamma(V)$, $f \mapsto f \circ \tau_g$, so we have a (right) action of G on $\Gamma(V)$.

Lemma C.1 The ring of invariants

$$\Gamma(V)^G = \{ f \in \Gamma(V) \mid \tau_g^*(f) = f \ \forall g \in G \}$$

is a finitely generated K-alg.

Def Let $\Gamma(V)^G \cong K[Y_1, \dots, Y_m]/I$. Then, we define the quotient variety

$$V^G := V(I) \subseteq \mathbb{A}^m_K \quad (\text{with } \Gamma(V^G) = \Gamma(V)^G)$$

and the <u>quotient morphism</u>

$\pi : V \to V^G$ is the morphism corr. to

the inclusion $\pi^*: \Gamma(V^G) = \Gamma(V)^G \hookrightarrow \Gamma(V)$.

<u>Note</u> By def., $\underset{\substack{\shortparallel \\ \pi\circ\tau_g(x)}}{\pi(gx)} = \pi(x)\ \forall g \in G, x \in V$.

(because $\tau_g^* \circ \pi^* = \pi^*$)

<u>Ex</u> Let the symm. gr. S_n act on $V = \mathbb{A}^n_K$

by permuting coordinates.

It acts on $\Gamma(V) = K[X_1, \dots, X_n]$ by

permuting variables.

$$\Gamma(V)^{S_n} = K[X_1, \dots, X_n]^{S_n} = \text{ring of symm. pol.}$$

$$K[X_1, \dots, X_n]^{S_n} \cong K[Y_1, \dots, Y_n]$$

i-th elem. symm. pol. $\leftarrow\!\shortmid$ Y_i

$X_1 + \dots + X_n \leftarrow\!\shortmid Y_1$

$X_1X_2 + X_1X_3 + \dots + X_{n-1}X_n \leftarrow\!\shortmid Y_2$

$X_1 \cdots X_n \leftarrow\!\shortmid Y_n$

$\rightsquigarrow (\mathbb{A}^n_k)^{S_n} = \mathbb{A}^n_K$

$\pi : \mathbb{A}^n_k \longrightarrow \mathbb{A}^n_K$

$(x_1, \dots, x_n) \longmapsto (y_1, \dots, y_n)$ where

y_i is the i-th elem. symm. pol. in $x_1, \dots, x_n$.

Pf of Lemma

Since $V \subseteq \mathbb{A}^n_k$, $\Gamma(V)$ is fin. gen.

W.l.o.g., the set of generators is closed under the action of G.

Say $\Gamma(V) = k[X_1, \dots, X_r]/I$ and G acts on $X_1, \dots, X_r$ by permutation.

(so we have an embedding $V \subseteq \mathbb{A}^r_k$ such that the action of G on V is the restriction of a permutation action on the coord. in $\mathbb{A}^r_k$.)

Consider the quotient hom.

$$k[X_1, \dots, X_r]^G \longrightarrow (k[X_1, \dots, X_r]/I)^G.$$

It is <u>surjective</u>:

Let $f \in K[X_1, \dots, X_r]$ such that
$\tau_g^*(f) \equiv f \mod I \quad \forall g \in G$.
Then, the "orbit average":

$$\frac{1}{|G|} \sum_{g \in G} \tau_g^*(f) \text{ lies in } K[X_1, \dots, X_r]^G$$

and is $\equiv f \mod I$.

But $K[X_1, \dots, X_r]^{S_r} \subseteq K[X_1, \dots, X_r]^G \subseteq K[X_1, \dots, X_r]$.
LHS is a fin. gen. K-alg. and
RHS is a fin. gen. LHS-mod.

$\Rightarrow K[X_1, \dots, X_r]^G$ is a fin. gen. K-alg.

$\Rightarrow \Gamma(V)^G = (K[X_1, \dots, X_r]/I)^G$ is a fin. gen. K-alg.

$\square$

<u>Ex</u> $G = \{\pm 1\}$ act on $V = \mathbb{A}_K^2$ by $(-1)\cdot(x,y) = (-x,-y)$.
We have $K[X,Y]^G = K[X^2, XY, Y^2]$
$\cong K[A,B,C]/(AC - B^2)$
(Syzygy)

$\rightsquigarrow V^G = \{(a,b,c) \in K^3 \mid ac = b^2\}$

$\pi : V \to V^G$
$(x,y) \mapsto (x^2, xy, y^2)$

<u>Lemma C.2</u> The map $\pi : V(\overline{K}) \to V^G(\overline{K})$ is surj. and each fiber is a G-orbit in $V(\overline{K})$.

<u>Cor C.3</u> $\dim(V^G) = \dim(V)$.

<u>Ese</u> $S_n \circlearrowright \mathbb{A}^n_K$

$\rightsquigarrow$ The preimage of $(y_1, \dots, y_n) \in \overline{K}^n$ is the set of $(x_1, \dots, x_n) \in \overline{K}^n$ such that

$$(T - x_1) \cdots (T - x_n) = T^n - y_1 T^{n-1} + \dots + (-1)^n y_n$$

(the root-with-correct-multiplicity-tuples).

<u>Rmk</u> ~~Im~~part. $\pi : V(K) \to V^G(K)$ is in general <u>not</u> surjective.

<u>Pf of Lemma</u> W.l.o.g. $K = \overline{K}$.

<u>surj</u> Let $Q \in V^G(K)$ and let $m_Q \subseteq \Gamma(V^G)$ be the corr. max. ideal. We want to show that

$$\emptyset \neq \{ P \in V(\overline{K}) \mid \pi(P) = Q \} = \{ P \in V(\overline{K}) \mid \underset{\pi^*(f)(P)}{\overset{\|}{f(\pi(P))}} = 0 \ \forall f \in m_Q \}.$$

(π^* is the inclusion map $\Gamma(V)^G \to \Gamma(V)$)

By Hilbert's Nsts, this is equivalent to

$$1 \notin (\text{id. of } \Gamma(V) \text{ gen. by } \cancel{\pi^*}(m_Q)).$$

Say $1 = \sum_i h_i f_i$ with $h_i \in \Gamma(V)$,

$f_i \in m_Q \subseteq \Gamma(V)^G$.

$$\Rightarrow 1 = \frac{1}{|G|} \sum_{g \in G} \sum_i \underbrace{\tau_g^*(h_i f_i)}_{\tau_g^*(h_i) f_i}$$

$$= \frac{1}{|G|} \sum_i \underbrace{\left(\sum_g \tau_g^*(h_i) \right)}_{\in \Gamma(V)^G} \underbrace{f_i}_{\substack{\in m_Q \\ (\text{id. of } \Gamma(V)^G)}} \in m_Q$$

↯

<u>fibers are G-orbits</u> Let $P_1, P_2 \in V(\bar{k})$.

If $P_1 \notin \{ g P_2 \mid g \in G \}$, by the Chin. remainder theorem, there is a fct. $h \in \Gamma(V)$ with $h(P_1) = 0$, $h(g P_2) = 1 \ \forall g \in G$.

Consider $f := \prod_{g \in G} \tau_g^*(h) \in \Gamma(V)^G$.

We have $f(P_1) = 0$, $f(P_2) = 1 \neq 0$.

$\Rightarrow \pi(P_1) \neq \pi(P_2)$ because $f = \pi^*(f)$. $\square$

Def Let $V \subseteq \mathbb{A}^n_k$ and $r \geq 1$. The <u>r-th symmetric power</u> of V is

$$V^{(r)} := (\underbrace{V \times \ldots \times V}_{r})^{S_r}.$$

Lemma C.3 Assume $C/\mathbb{A}^n_k$ is a smooth curve. Then, $C^{(r)}$ is smooth (of dim. r).

Rmk $((\mathbb{A}^n_k)^r)^{S_r}$ is singular for $n, r \geq 2$. (?)

(at the image of $(0, \ldots, 0) \in (\mathbb{A}^n)^r$ in $((\mathbb{A}^n)^r)^{S_r}$)

Pf (sketch) w.l.o.g. $k = \overline{k}$.

Consider a point $(P, \ldots, P) \in C \times \ldots \times C$.

Let Q be its image in $C^{(r)}$.

The local ring $\mathcal{O}_{C,P}$ is a DVR with residue field $k = \overline{k}$ and uniformizer $t_{C,P}$.

Let $\widehat{\mathcal{O}}_{C,P}$ be its completion with max. ideal $\widehat{m}_{C,P} = m_{C,P}\widehat{\mathcal{O}}_{C,P}$.

We obtain an isomorphism $k[[T]] \to \widehat{\mathcal{O}}_{C,P}$.

$$T \mapsto t_{C,P}$$

("Analytically, any curve looks like $\mathbb{A}^1$ near a smooth point.")

The completion of $\mathcal{O}_{C^{(r)},Q}$ (at $m_{C^{(r)},Q}$)

is $\hat{\mathcal{O}}_{C^{(r)},Q} = \varprojlim_{n\to\infty} \mathcal{O}_{C^{(r)},Q} / m^n_{C^{(r)},Q}$

$$\cong K[[T_1, \dots, T_r]]^{S_r}$$

$$\cong K[[\underbrace{U_1, \dots, U_r}_{\substack{\text{elem. symm.} \\ \text{pol. in } T_1,\dots,T_r}}]]$$

The max. ideal $\hat{m}_{C^{(r)},Q}$ of this ring is $(U_1, \dots, U_r)$. It satisfies

$$\begin{array}{ccc} \hat{m}_{C^{(r)},Q} / \hat{m}^2_{C^{(r)},Q} & \cong & \hat{m}_{(\mathbb{A}^1)^{(r)},0} / \hat{m}^2_{(\mathbb{A}^1)^{(r)},0} \\ \| & & \| \\ m_{C^{(r)},Q} / m^2_{C^{(r)},Q} & & m_{(\mathbb{A}^1)^r,0} / m^2_{(\mathbb{A}^1)^r,0} \\ & & \cong \\ & & K^r \end{array}$$

$\Rightarrow$ dim (cotangent space at Q) $= r = \dim(C^{(r)})$.

("Same" for other tuples $(P_1, \dots, P_r)$ ...) $\square$

Rmk You can perform a similar construction for $V \subseteq \mathbb{P}^n_K$.

Rmk If E is an ell. curve (def. over K) and $T \subseteq E(\overline{K})$ is a finite subgroup, this allows us to construct a quotient E/T, which is again an ell. curve with isogeny $E \to E/T$.

If $T \subseteq E$ is def. over K, then E/T is def. over K.