Thm 3.3.10 (Rigidity theorem 2)

Let V, W be smooth varieties, A an abelian variety, $V \times W$ geom. irred. Let $\alpha : V \times W \to A$ be a morphism such that

$$\alpha(V \times \{w_0\}) = \alpha(\{v_0\} \times W) = \{a_0\}$$

for some $v_0 \in V(K)$, $w_0 \in W(K)$, $a_0 \in A(K)$.

Then, $\alpha(V \times W) = \{a_0\}$.

Pf W.l.o.g. K is alg. closed and $V \subseteq \mathbb{A}^n_K$.

If $V = C$ is a curve:

Let $C' \supseteq C$ be a smooth proj. curve.

By Thm 3.3.7, $\alpha : C \times W \to A$ extends to $\alpha' : C' \times W \to A$

By continuity, we still have

$$\alpha'(C' \times \{w_0\}) = \alpha'(\{v_0\} \times W) = \{a_0\}.$$

Since C' is complete, we can then apply the original rigidity theorem.

For any V: Consider an irreducible curve $v_0 \in C \subseteq V$ which is smooth at v_0. Let $\pi : C' \to C$ be a normalisation. It induces a morphism

$$\alpha' : C' \times W \to A \text{ with}$$

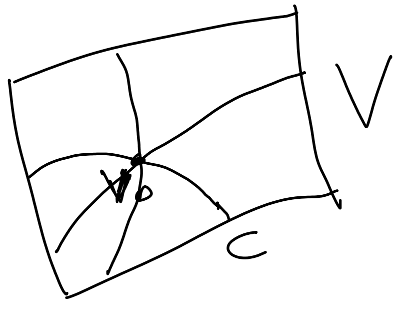
$$\alpha'(C' \times \{w_0\}) = \alpha'(\{\pi^{-1}(v_0)\} \times W) = \{a_0\}$$

a single point because C is nonsingular at v_0

We saw above that this implies

$$\alpha'(C' \times W) = \{a_0\}.$$

Then, the claim follows by continuity from the following Lemma. $\square$



Lemma 3.3.11 Let k be alg. closed, $V \subseteq \mathbb{A}^n_k$ irred., $v_0 \in V(k)$ a nonsingular point. Then, the union of the irred. curves $v_0 \in C \subseteq V$ which are nonsingular at v_0 is (Zariski) dense in V.

Cor 3.3.12 Let V, W be smooth, $v_0 \in V(K)$, $w_0 \in W(K)$, $V \times W$ geom. irred., A an abelian variety, $\alpha : V \times W \to A$ a morphism with $\alpha(v_0, w_0) = 0$. Then, there are (unique) morphisms $\varphi : V \to A$, $\psi : W \to A$ such that $\alpha(v, w) = \varphi(v) + \psi(w)$ for all $v \in V(\overline{K})$, $w \in W(\overline{K})$ and $\varphi(v_0) = \psi(w_0) = 0$.

Pf We need to take

$$\varphi(v) = \alpha(v, w_0), \quad \psi(w) = \alpha(v_0, w).$$

Then, $\alpha'(v, w) := \alpha(v, w) - \varphi(v) - \psi(w)$ satisfies the assumptions of the rigidity theorem 2. $\Rightarrow \alpha'(V \times W) = \{0\}$. $\square$

Cor 3.3.13 Let $\varphi : G \to A$ be a morphism of varieties from a connected group var G to an abelian var. A sending the identity $e \in G$ to identity $0 \in A$. Then, φ is a group homomorphism.

Pf Same as Thm 3.3.2 but using rigidity theorem 2. $\square$

Cor 3.3.14 Any morphism $\varphi: \mathbb{A}^n_k \to A$ to an abelian variety is constant.

Pf $n=1$: The morphism

$$\mathbb{A}^1_k = \mathbb{G}_a \longrightarrow A$$
$$x \longmapsto \varphi(x) - \varphi(0)$$

is a group homomorphism.

The morphism

$$\mathbb{A}^1_k \supset \mathbb{G}_m \longrightarrow A$$
$$(x, x^{-1}) \longmapsto \varphi(x) - \varphi(1)$$

is a group homomorphism.

$$\Rightarrow \varphi(x+y) + \varphi(0) = \varphi(x) + \varphi(y) \quad \forall x,y \in \bar{k}$$
$$\varphi(xy) + \varphi(1) = \varphi(x) + \varphi(y) \quad \forall x,y \in \bar{k}^\times$$

$$\Rightarrow \varphi(x+y) + \varphi(0) = \varphi(xy) + \varphi(1) \quad \forall x,y \in \bar{k}^\times$$

Pick $y = -x$.

$$\Rightarrow 2\varphi(0) = \varphi(-x^2) + \varphi(1) \quad \forall x \in \bar{k}^\times$$

$$\Rightarrow 2\varphi(0) = \varphi(x) + \varphi(1) \quad \forall x \in \bar{k}^\times$$

$$\Rightarrow \varphi(x) = \text{const.}$$

$n>1$: Use induction and Cor 3.3.12. $\square$

3.4. The Jacobian variety

Let C be a smooth proj. curve over K of genus g with $C(K) \neq \emptyset$.

Goal: Construct an abelian variety $J = J_C$ (the Jacobian variety of C) such that we have a group isomorphism

$$J(K) = \mathcal{Cl}^0(C).$$

Ex If C is an ell. curve, we have previously considered the bijection

$$C(K) \xrightarrow{\sim} \mathcal{Cl}^0(C)$$
$$P \longmapsto [P] - [O]$$

$\rightsquigarrow$ The Jacobian variety of an ell. curve E is $J_E = E$.

Ex If $C = \mathbb{P}^1_K$, we have shown $\mathcal{Cl}^0(C) = 1$.

$\rightsquigarrow$ The Jacobian variety of $\mathbb{P}^1_K$ is the trivial abelian variety 1.

Rmk In fact, we want a group isom.

$J(L) = Cl^0(C_L)$ for every field ext. $L|K$, where C_L is the same curve C, but with base field L. The isom. should commute:

$$\begin{array}{ccc} J(L) & = & Cl^0(C_L) \\ \downarrow & & \downarrow \\ J(L') & = & Cl^0(C_{L'}) \end{array} \qquad \begin{array}{c} L' \\ | \\ L \\ | \\ K \end{array}$$

Rmk If $C(K) \neq \emptyset$ and $L|K$ is a Gal. ext., we have

$$Cl^0(C) = Cl^0(C_L)^{Gal(L|K)}.$$

Rmk We will obtain a map

$$\begin{array}{ccccc} C(K) & \longrightarrow & Cl^0(C) & = J(K) \\ P & \longmapsto & [P]-[Q] & \end{array}$$

for any fixed $Q \in C(K)$.

$C \hookrightarrow J$ is injective if $g \geq 1$.

Idea of construction of J

Fix some point $P \in C(K)$. Then,

$$\underbrace{C(\overline{K}) \times \dots \times C(\overline{K})}_{g} / S_g \longrightarrow \mathcal{C}\ell^{\circ}(C_{\overline{K}})$$

$$(Q_1, \dots, Q_g) \longmapsto Q_1 + \dots + Q_g - gP$$

is "almost a bijection", where S_g denotes the symmetric group of order $g!$ (acting on $C \times \dots \times C$ by permutation).

Surjective: Let $D \in \mathcal{C}\ell^{\circ}(C_{\overline{K}})$.

$\Rightarrow \deg(D + gP) = g$

$\underset{R-R}{\Rightarrow} \ell(D + gP) \geq 1$

$\Rightarrow$ $D + gP$ lies in the same divisor class as some divisor $E \geq 0$ (of degree g). Write $E = Q_1 + \dots + Q_g$.

Almost injective:

For "generic" $D \in \mathrm{Div}^0(C_{\bar{k}})$, we have only one preimage $(Q_1, \dots, Q_g)$ because $\ell(D + gP) = 1$:

By R-R, $\ell(2gP) = g+1$.

By Cor 1.10,

$\ell(2gP - R_1) = g$ for a.a. $R_1 \in C(\bar{k})$.

For any such R_1, by Cor 1.10,

$\ell(2gP - R_1 - R_2) = g-1$ for a.a. $R_2 \in C(\bar{k})$.

$\vdots$

$\ell(\underbrace{2gP - R_1 - \dots - R_g}_{\text{div of deg } g}) = 1$ for a.a. $R_g \in C(\bar{k})$

$$\Rightarrow \emptyset \neq \{(R_1, \dots, R_g) \mid P, R_1, \dots, R_g \text{ distinct}, \ \ell(2gP - R_1 - \dots - R_g) = 1\} =: T$$

Claim: This set T is an open subset of $C \times \dots \times C$.

Pf Let $P, R_1, \dots, R_g$ be distinct. Then,

$2gP - R_1 - \dots - R_g + \mathrm{div}(h) \geq 0 \Leftrightarrow 2gP + \mathrm{div}(h) \geq 0,$
$h(R_1) = \dots = h(R_g) = 0.$

Hence, if $f_1, \dots, f_{g+1}$ form a basis of $L(2gP)$, then $(R_1, \dots, R_g) \in T$ if and only if the matrix $(f_i(R_j))_{ij}$ has rank g. $\square$

<u>Steps</u>:

1) Pick $\emptyset \neq U \subseteq C$ open and affine s.t. $U \times \dots \times U / S_g \longrightarrow \mathcal{C}\ell^0(C_{\bar{K}})$ is injective.

2) Construct an affine variety $U^{(g)}$ whose pts are in bijection with elements of $U \times \dots \times U / S_g$.

3) Show that the group op. $+$ on $\mathcal{C}\ell^0(C_{\bar{K}})$ is described by a rational map $U^{(g)} \times U^{(g)} \dashrightarrow U^{(g)}$.

4) Cover $\mathcal{C}\ell^0(C_{\bar{K}})$ by translates of the image of $U^{(g)}$. These translates will form affine charts of J_C.

5) Show that the variety J_C is <u>complete</u>.