3.3. Basic properties of abelian varieties

<u>Def</u> An <u>abelian variety</u> is a complete irreducible group variety.

<u>Ex</u> Elliptic curves

<u>Non-ex</u> $\mathbb{G}_a$, $\mathbb{G}_m$ (!!!)

$\mathbb{G}_a = \mathbb{A}^1_K$

<u>Thm 3.3.1</u> Any abelian variety A is commutative (!)

<u>Pf</u> $T = \{(x, yxy^{-1}) \mid x, y \in A\}$ is the image of

$$A \times A \longrightarrow A \times A, \quad (x,y) \longmapsto (x, yxy^{-1})$$

and hence closed (because A is complete) and irreducible (because A is (geom.) irreducible).

Also, T contains exactly one point of the form (e, a) with $a \in A$, namely (e,e).

$\Rightarrow$ The preimage of e under the proj. $T \to A$, $(s,t) \mapsto s$ is just the point (e,e).

$$\Rightarrow \quad \dim(T) \leq \dim(A).$$

$\dim(\pi^{-1}(y)) \geq \dim(X) - \dim(Y)$
for any $\pi : X \to Y$
and $y \in \pi(X)$

On the other hand, $\underbrace{\{(x,x) \mid x \in A\}}_{\dim(\cdot) = \dim(A)} \subseteq \underset{\text{irred.}}{T}$.

$$\Rightarrow \{(x,x) \mid x \in A\} = T.$$

$$\Rightarrow y x y^{-1} = x \ \forall x, y \in A. \qquad \square$$

$\rightsquigarrow$ We write ab. var. additively.

Thm 3.3.2 Let A, B be abelian varieties. Then, any morphism $\varphi : A \to B$ sending $0 \in A$ to $0 \in B$ is a group hom.

Pf Consider the morphism

$$\alpha : A \times A \longrightarrow B$$
$$(a_1, a_2) \longmapsto \varphi(a_1 + a_2) - \varphi(a_1) - \varphi(a_2)$$

We have $\alpha(A \times \{0\}) = \{0\} = \alpha(\{0\} \times A)$.
Since A is complete and (geometrically) irreducible, Thm 3.1.6 (Rigidity) shows
$\alpha(A \times A) = \{0\}$. $\square$

Cor 3.3.3 The group operation on an abelian variety A is determined by the variety A and the identity element $O \in A(k)$.

Pf The identity morphism $id : A \to A$ is a group hom. for any ab. var. structures on the LHS and RHS. $\square$

Rmk The Thm is wrong for general group var. A, B:

E.g. $\mathbb{G}_m \hookrightarrow \mathbb{G}_a$ is not a group hom.

$x = (x, x^{-1}) \longmapsto x$

Rmk The Thm is correct if A is ~~an~~ ab. var. and B is any group var.

(The image of A is an ab. var.)

Rmk The Thm is correct if B is an ab. var. and A is any group var. (need another version of the rigidity thm ...)

Lemma 3.3.4 Let V be a smooth (or just normal) variety, W be a complete variety. Then, any rational map $\varphi : V \dashrightarrow W$ is defined on (= can be extended to) an open subset $U \subseteq V$ with $\dim(V \setminus U) \leq \dim(V) - 2$.

Exe

$$\mathbb{A}^2_K \dashrightarrow \mathbb{P}^1_K \quad \text{is def. everywhere}$$
$$(x, y) \longmapsto [x : y] \quad \text{except at } O.$$

Rmk Completeness required:

$$\mathbb{A}^1_K \dashrightarrow \mathbb{A}^1_K \quad \text{can't be extended to } O$$
$$x \longmapsto \frac{1}{x}$$

Rmk Smoothness (or normality) required

$$\{(x, y) \in \mathbb{A}^2_K \mid x^3 = y^2\} \dashrightarrow \mathbb{P}^1_K$$
$$(x, y) \longmapsto [x : y]$$

can't be extended to O

(The composition

$$\mathbb{A}^1_K \to \{\ldots\} \to \mathbb{P}^1_K$$
$$t \mapsto (t^2, t^3) \underset{\substack{\uparrow \\ \text{for } t \neq 0}}{\mapsto} [t^2 : t^3] = [1 : t]$$

would by continuity send every t to $[1 : t]$, so it's "the identity" $\mathbb{A}^1 \to \mathbb{A}^1$. But its derivative at $t = 0$ is 0.)

Cor 3.3.5 If V is a smooth curve and W is complete, then any rat. map $V \dashrightarrow W$ is (= can be extended to) a morphism $V \to W$.

Lemma 3.3.6 Let V be a smooth var, G be a group variety, and let $\varphi : V \dashrightarrow G$ be a rational map defined on an open set $U \subseteq V$ (which can't be extended to a larger open subset). Then, every irred. comp. of $V \setminus U$ has codimension 1 in V.

Pf Consider the rational map

$$\alpha : V \times V \dashrightarrow G$$
$$(x,y) \longmapsto \varphi(x)\varphi(y)^{-1}$$

Let α be defined on the open set $S \subseteq V \times V$ (and not extendable to any larger open set).

Claim: $x \in U \Leftrightarrow (x,x) \in S$

Pf: "$\Rightarrow$" Clear. In fact, $U \times U \subseteq S$.

"$\Leftarrow$" Since V is irred., the nonempty open subsets $U \subseteq V$ and $\{y \in V \mid (x,y) \in S\} \subseteq V$ intersect. Let $y \in U$ with $(x,y) \in S$.

Consider the open set

$U' = \{x' \in V \mid (x',y) \in S\}$ containing x

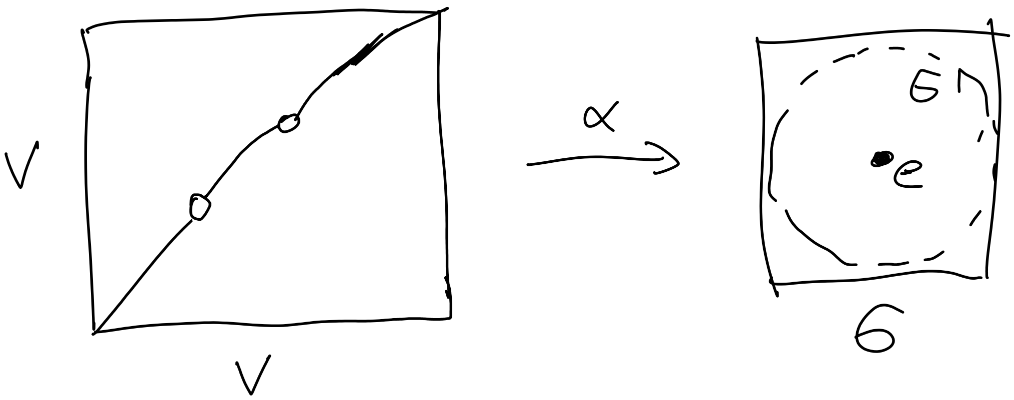
and the morphism

$$\varphi' : U' \longrightarrow G .$$
$$x' \longmapsto \alpha(x', y)\varphi(y)$$

Note that φ and φ' agree on $U \cap U' \subseteq V$.

$\Rightarrow$ φ can be extended to the open neighborhood $U \cup U'$ of x. $\Rightarrow x \in U$. $\square$

If α is def. at (x, x), then $\alpha(x,x) = \varphi(x)\varphi(x)^{-1} = e$.



Let $G' \subseteq G$ be an (open) affine patch containing e. Let $G' \subseteq \mathbb{A}^n_K$ be an embedding.

We obtain a rat. map $\alpha' : V \times V \dashrightarrow G' \subseteq \mathbb{A}^n_K$ given by rat. fcts. $\alpha'_1, \dots, \alpha'_n \in K(V \times V)$.

Consider the divisor $D_i = \operatorname{div}(\alpha'_i) \subseteq V \times V$.

Write $D_i = \sum\limits_{\substack{Z \subseteq V \\ \text{irred.} \\ \text{codim } 1}} c_{i,Z} Z$.

φ def. at x

$\Leftrightarrow \alpha$ def. at (x,x)

$\Leftrightarrow \alpha'$ def. at (x,x)

$$\Leftrightarrow (x,x) \notin \bigcup_i \underbrace{\bigcup_{Z : c_{i,Z} < 0} Z}_{\substack{\text{pole divisor} \\ \text{of } \alpha_i'}}$$

For any $Z \subseteq V \times V$ irred. of codimension 1 as above, each irred. comp. of

$$\{x \in V \mid (x,x) \in Z\} \subseteq V$$

has codimension ≤ 1 in V. $\square$

<u>Thm 3.3.7</u> Let V be a smooth variety and A an abelian variety. Then, any rat. map $V \dashrightarrow A$ is (= can be extended to) a morphism $V \longrightarrow A$.

<u>Pf</u> By Lemma 3.3.5, it is def. everywhere or undef. on a set of codim. 1.

By Lemma 3.3.4, it's undef. at most on a set of codim ≥ 2. $\square$

<u>Lemma 3.3.8</u> Any smooth curve C is an open subset of a (unique) smooth projective curve C'.

<u>Lemma 3.3.9</u> For any irred. curve C with smooth locus $U \subseteq C$, there is a smooth curve C' (the <u>normalization</u> of C) with a morphism $\pi : C' \to C$ which induces an isomorphism $U' \to U$ for $U' = \pi^{-1}(U)$.

<u>Pf</u> See Chapter I. 6 in Hartshorne. $\square$

<u>Exe</u> $\mathbb{A}^1_K \longrightarrow \{(x,y) \mid x^3 = y^2\}$

$t \longmapsto (t^2, t^3)$

<u>Rmk</u> If $C \subseteq \mathbb{A}^n_K$ is an affine curve, let R be the integral closure of $\Gamma(C)$ in $K(C)$. It's a fin. gen. ring ext. of K, say $R = K[f_1, \dots, f_m]$. Let I be the kernel of $K[X_1, \dots, X_m] \to R$.

$X_i \mapsto f_i$

$\Gamma(C) \subseteq R = K[X_1, \dots, X_m]/I$. Take $C' = V(I) \subseteq \mathbb{A}^m_K$,

$C' \to C$ corr. to the inclusion $\Gamma(C) \hookrightarrow R$.

Ex $C = \{(x,y) \mid y^2 = x^2(x+1)\}$

$$\Gamma(C) = K[X,Y]/(Y^2 - X^2(X+1))$$

$$R = \Gamma(C)[\underbrace{\tfrac{Y}{X}}_{Z}] = K[X,Z]/(Z^2-(X+1))$$

$$C' = \{(x,z) \mid z^2 = x+1\}$$

$$C' \longrightarrow C$$

$$(x,z) \longmapsto (x,xz)$$