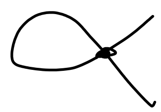
Some AG

Def An affine variety $V \subseteq \mathbb{A}^n_K$ is <u>normal</u> if it is irreducible and $\Gamma(V)$ is integrally closed in its field of fractions $K(V)$.

A projective variety $V \subseteq \mathbb{P}^n_K$ is <u>normal</u> if it is irreducible and its nonempty affine charts $V \cap H_i$ are normal.

<u>Ex</u> $\mathbb{A}^n$, $\mathbb{P}^n$, any smooth variety

<u>Non-ex</u> $V = \{(x,y) \mid x^2 = y^2(y+1)\} \subseteq \mathbb{A}^2_K$



$\frac{x}{y} \in K(V)$ is integral over $\Gamma(V)$, but not contained in $\Gamma(V)$.

$$\left(\frac{x}{y}\right)^2 = y+1$$

<u>Def</u> A morphism $\varphi: V \to W$ between affine varieties V, W is <u>finite</u> if $\Gamma(V)$ is an integral ring ext. of $\varphi^*(\Gamma(W))$.

A morphism $\varphi: V \to W$ between projective varieties V, W is <u>finite</u> if its restrictions to affine charts are finite

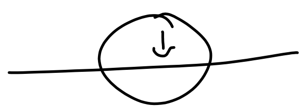
$$(\varphi^{-1}(W \cap H_j) \cap H_i \longrightarrow W \cap H_j).$$

Ex - Any inclusion

- $\varphi : \{(x,y) \in \mathbb{A}^2_k \mid x^2+y^2=1\} \longrightarrow \mathbb{A}^1_k$

$$(x,y) \longmapsto x$$

because $\Gamma(V) = k[X][Y]/(Y^2+(X^2-1))$ is an integral ring ext of $\Gamma(W) = k[X]$.



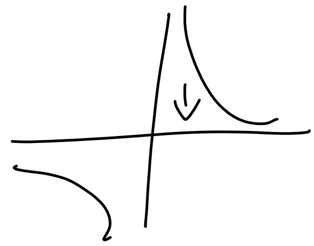
Non-ex - $\mathbb{A}^2_k \longrightarrow \mathbb{A}^1_k$

$$(x,y) \longmapsto x$$

- $\varphi : \{(x,y) \in \mathbb{A}^2_k \mid xy=1\} \longrightarrow \mathbb{A}^1_k$

$$(x,y) \longmapsto x$$

because $\Gamma(V) = k[X][Y]/(XY-1)$ is not an integral ring ext of $\Gamma(W) = k[X]$.



Rmk All fibers $\varphi^{-1}(P)$ $(P \in W(\overline{k}))$ of finite morphisms are finite.

Rmk Any dominant finite morphism is surjective (over $\overline{k}$).

Rmk Compositions of finite morphisms are finite.

Cor Restrictions of finite morphisms are finite.

Rmk If $\varphi: V \to W$ and $W \subseteq W'$, then $\varphi: V \to W$ is finite if and only if $\varphi: V \to W'$.

Cor Any finite morphism is closed: the image of any (Zariski) closed set is closed.

Rmk Finite morphisms preserve dimension: $\dim(\varphi(X)) = \dim(X)$.

Rmk Let $\varphi: V \to W$ be a morphism between smooth projective varieties. Then, φ is finite if and only if $\varphi^{-1}(P) \subseteq V(\overline{K})$ is finite for all $P \in W(\overline{K})$.

Ex Any nonconstant morphism between smooth projective curves is finite.

Def Let $\varphi : V \to W$ be dominant finite morphism between affine normal varieties, its **degree** $n = \deg(\varphi) = [K(V) : K(W)]$.

Rmk $\Gamma(V)$ is the integral closure of $\varphi^*(\Gamma(W))$ in $K(V)$.

We consider $\Gamma(W)$ a subring of $\Gamma(V)$ with the inclusion map $\varphi^* : \Gamma(W) \to \Gamma(V)$.

Def The **discriminant** $\operatorname{disc}(\Gamma(V) | \Gamma(W))$ is the ideal of $\Gamma(W)$ generated by the determinants $\det((\operatorname{Tr}_{K(V)|K(W)}(f_i f_j))_{i,j}) \in \Gamma(W)$ with $f_1, \ldots, f_n \in \Gamma(V)$.

Rmk Like in number theory, the discriminant determines ramification: For $S \subseteq V$, $T \subseteq W$ irred. of codim. 1, consider the ram. index

$$e_{S|T} = v_{V,S}(t_{W,T} \circ \varphi).$$

φ is unramified at T (meaning $e_{S|T} = 1 \ \forall S$) if and only if $\operatorname{disc}(\Gamma(V) | \Gamma(W)) \in \Gamma(W)$ doesn't vanish on T (meaning $v_{W,T}(\operatorname{disc}) = 0$).

Rmk φ is unramified if and only if every $Q \in W(\overline{K})$ has exactly n preimages $P \in V(\overline{K})$.

Rmk Assume φ is unramified.

Let $Q \in W(K)$ and let $m_Q \subset \Gamma(W)$ be the corr. maximal ideal.

Note that $P \in V(\overline{K})$ lies in $\varphi^{-1}(Q)$ if and only if

$$\underbrace{f(\varphi(P))}_{= \varphi^*(f)(P)} = 0 \quad \forall f \in m_Q .$$

$\Rightarrow \varphi^{-1}(Q) \subseteq V(\overline{K})$ is the vanishing locus of the ideal $(\varphi^*(m_Q))$ of $\Gamma(V)$.

Then,

$$\Gamma(V)/(\varphi^*(m_Q)) = \Gamma(\varphi^{-1}(Q)) = \prod_{\substack{R \subseteq \varphi^{-1}(Q) \\ \text{0-dim.} \\ \text{subvar.}}} \Gamma(R)$$

$\Gamma(\{(\pm i, 1)\}) = \Gamma(\{\pm i\}) = \mathbb{Q}[x]/(x^2+1) = \mathbb{Q}(i)$

$$= \prod_{\substack{\mathrm{Gal}(\overline{K}|K)\text{-orbit} \\ \text{of points } P \in \varphi^{-1}(Q)}} (\text{field of def. } K' \text{ of } P)$$

We will prove a slightly stronger version of Thm 2.8.1.:

<u>Thm 2.8.2</u> Let V, W be affine normal varieties over a number field K and let $\varphi : V^{\subseteq \mathbb{A}^n} \to W^{\subseteq \mathbb{A}^m}$ be a dominant finite unramified morphism. Then, there is a finite set S of primes of K such that the field of definition K' of any point $P \in V(\overline{K})$ with $Q := \varphi(P) \in W(K)$ is unramified at all primes $\mathfrak{q} \notin S$ for which $v_{\mathfrak{q}}(y_i) \geq 0$ for all coordinates y_i of Q.

<u>Pf</u> (see Lang, Fundamentals of Diophantine Geometry, Chapter 2.8)

$$\begin{array}{ccccc} K(V) & \supset & \Gamma(V) & \supset & B \\ \varphi^* \big\uparrow & & \big\uparrow & & \big\uparrow \\ K(W) & \supset & \Gamma(W) & \supset & A \end{array}$$

Consider the discriminant ideal

$$\mathrm{disc}(\Gamma(V) \mid \Gamma(W)) \subseteq \Gamma(W).$$

Since φ is unramified, its vanishing locus is $\emptyset$.

By Hilbert's Nullstellensatz, this means that disc $(\Gamma(V)|\Gamma(W)) = \Gamma(W)$.

Consider polynomials $f_1, \dots, f_m \in K[X_1, \dots, X_n]$ defining the morphism φ. Let S_1 be the set of primes of K such that some coeff. of some f_i has negative $\mathfrak{p}$-adic valuation.

Let $\mathcal{O}_K \subseteq \mathcal{O}_{S_1} < K$ be the ring of S_1-integers (the ring of $g \in K$ with $v_{\mathfrak{p}}(g) \geq 0 \ \forall \mathfrak{p} \notin S_1$).

$\Rightarrow f_1, \dots, f_m \in \mathcal{O}_{S_1}[X_1, \dots, X_n]$.

We obtain rings

$$A = \mathcal{O}_{S_1}[Y_1, \dots, Y_m] / (I(W) \cap \mathcal{O}_{S_1}[Y_1, \dots, Y_m])$$

$$B = \mathcal{O}_{S_1}[X_1, \dots, X_n] / (I(V) \cap \mathcal{O}_{S_1}[X_1, \dots, X_n]).$$

$$A \otimes_{\mathcal{O}_{S_1}} K = \Gamma(W), \quad B \otimes K = \Gamma(V)$$

A is integrally closed in $\Gamma(W)$

B is " — $\Gamma(V)$.

We have $\left(\mathrm{disc}(B|A)\right)_{\Gamma(W)} = \mathrm{disc}(\Gamma(V)|\Gamma(W)) = \Gamma(W)$.

$\Rightarrow$ $\mathrm{disc}(B|A) \subseteq A$ contains a nonzero constant $0 \neq c \in \mathcal{O}_{S_1}$.

Let $S := S_1 \cup \{\mathfrak{q} \mid v_{\mathfrak{q}}(c) > 0\}$.

Now, take $Q \in W(K)$ and $m_Q \subset \Gamma(W)$ the corr. max. ideal and $m_Q' = m_Q \cap A$.

$$A_Q := A/m_Q' \cong \mathcal{O}_{S_1}(Y_1, \dots, Y_m)/(I(W), Y_1 - q_1, \dots, Y_m - q_m)$$
$$\cong \mathcal{O}_{S_1} \text{ if } q_1, \dots, q_m \in \mathcal{O}_{S_1}.$$

$$\Gamma(W)/m_Q \cong K$$

$$\Gamma(V)/(\varphi^*(m_Q)) = \prod_{\substack{\text{Gal-orbit} \\ \text{of } P \in \varphi^{-1}(Q)}} (\text{field of def. } K' \text{ of } P)$$

$$B_Q := B/(\varphi^*(m_Q')) = \prod_{\dots} (\text{ring of } S_1\text{-integers of field of def. } K' \text{ of } P$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\|$

int. cl. of $\mathcal{O}_{S_1}$ in K'

The discriminant ideal $\mathrm{disc}(B_{\mathfrak{q}}|A_{\mathfrak{q}})$ is the image of $\mathrm{disc}(B|A)$ under the map $A \longrightarrow A/m_{\mathfrak{q}}'$

$x \longmapsto x \bmod m_{\mathfrak{q}}'$

$\Rightarrow$ $0 \neq c \in \mathcal{O}_{S_1}$ lies in $\mathrm{disc}(B_{\mathfrak{q}}|A_{\mathfrak{q}})$

$$\mathrm{disc}(B_{\mathfrak{q}}|A_{\mathfrak{q}}) = \prod \mathrm{disc}(\text{int. cl. of } \mathcal{O}_{S_1} \text{ in } K' \mid \mathcal{O}_{S_1})$$

$\Rightarrow$ None of the discr. on the RHS are divisible by any $\mathfrak{q} \notin S$.

$\mathrm{disc}(\mathcal{O}_{K'}|\mathcal{O}_K)$ $\qquad$ $\mathrm{disc}(\;\ldots\;|\mathcal{O}_{S_1})$

$\Rightarrow$ $\mathcal{O}_{K'}|\mathcal{O}_K$ is unram. at all primes $\mathfrak{q} \notin S$.

"$\square$"