Cor A.2.9 Let $f_0, \dots, f_m \in K(X_0, \dots, X_n)$ be homogeneous of degree d and consider the map

$$\varphi : \mathbb{P}^n \setminus V(f_0, \dots, f_m) \longrightarrow \mathbb{P}^m$$
$$[x_0 : \dots : x_n] \longmapsto [f_0(x_0, \dots, x_n) : \dots : f_m(x_0, \dots, x_n)]$$

Then, $h(\varphi(P)) \underset{\varphi}{\lesssim} d \cdot h(P) \ \forall P \in \mathbb{P}^n(\overline{K}) \setminus V(f_0, \dots, f_m)$

Cor A.2.10 Let $V \subseteq \mathbb{P}^n_K$ be a projective variety and let $f_0, \dots, f_m \in K[X_0, \dots, X_n]$ be homogeneous of degree d and assume that $f_0, \dots, f_m$ have no common zeroes in $V(\overline{K})$, so that

$$\varphi : V \longrightarrow \mathbb{P}^m$$
$$[x_0 : \dots : x_n] \longmapsto [f_0(x_0, \dots, x_n) : \dots]$$

is well-defined.

Then, $h(\varphi(P)) \underset{\varphi}{\approx} d \cdot h(P)$ for all $P \in V(\overline{K})$.

A.3. Height and divisor classes

Thm A.3.1 Let C be a smooth projective curve. We can associate to every $D \in \mathrm{Div}(C)$ a function $h_D : C(\bar{K}) \to \mathbb{R}$ such that

i) $h_D(P) \underset{\varphi}{\approx} h(\varphi(P))$ for any morphism $\varphi : C \to \mathbb{P}^n$ defined by functions $f_0, \dots, f_n \in K(C)$ with associated divisor $D = \sum_Q n_Q Q$ and any $P \in C(\bar{K})$.

(D minimal s.t. $f_0, \dots, f_n \in L(D)$.

or: $n_Q = - \min_i v_Q(f_i)$)

ii) $h_{D+D'}(P) \underset{D,D'}{\approx} h_D(P) + h_{D'}(P) \quad \forall P \in C(\bar{K})$.

Rmk If D, D' lie in the same divisor class, then $h_D(P) \underset{D,D'}{\approx} h_{D'}(P)$ for any $P \in C(\bar{K})$.

Pf If $\varphi : C \to \mathbb{P}^n$ is def. by $f_0, \dots, f_n$ with divisor D, then φ is also def. by $f_0 g, \dots, f_n g$ with divisor $D + \mathrm{div}(g)$ for any $g \in K(C)^\times$. $\square$

Rmk h_D is unique up to bounded functions: If h'_D denotes other fcts as above, then

$$h_D(P) \underset{h_D, h'_D, D}{\approx} h'_D(P) \quad \forall D \in \mathrm{Div}(C),\ P \in C(\bar{K}).$$

Pf By Riemann-Roch, there is a morphism associated to any divisor of degree $\geq 2g_C$ (then, $L(D-P) = L(D) - 1\ \forall P$).

Any divisor D, can be written as $D = D_1 - D_2$ with $\deg(D_1), \deg(D_2) \geq 2g_C$.

$$\Rightarrow h_D(P) \underset{ii}{\approx} h_{D_1}(P) - h_{D_2}(P) \underset{i}{\approx} h(\varphi_1(P)) - h(\varphi_2(P))$$

$$\underset{i}{\approx} h'_{D_1}(P) - h'_{D_2}(P) \underset{ii}{\approx} h'_D(P),$$

where φ_i is any morphism associated to D_i.

$\square$

Rmk If $\deg(D)>0$, then

a) $h_D(P) \underset{D}{\gtrsim} 0 \quad \forall P \in C(\bar{K})$

b) For any finite field ext. $L|K$ and any $t \in \mathbb{R}$, there are only finitely many $P \in E(L)$ s.t. $h_D(P) \leq t$.

Pf For $n \geq 2g+1$, there is a closed embedding φ associated to nD, so

a) $n \cdot h_D(P) \underset{ii}{\approx} h_{nD}(P) \underset{i}{\approx} h(\underbrace{\varphi(P)}_{\in \mathbb{P}^m_K(\bar{K})}) \geq 0$

b) $h_D(P) \leq t \Rightarrow h_{nD}(P) \lesssim nt$
$$\begin{array}{c} \wr\wr \\ h(\underbrace{\varphi(P)}_{\in \mathbb{P}^m_K(L)}) \end{array}$$

By Thm A.2.1, there are only finitely many such $\varphi(P) \in \mathbb{P}^m_K(L)$ and hence finitely many such P (since φ is injective). $\square$

<u>Lemma A.3.2</u> If φ, φ' are defined by functions $f_0, \dots, f_n$ and $f_0', \dots, f_m'$ with the same divisor $D = \sum c_P P$, then

$$h(\varphi(P)) \underset{\varphi,\varphi'}{\approx} h(\varphi'(P)) \quad \forall P \in C(\bar{K}).$$

<u>Pf</u> W.l.o.g. (by transitivity), $f_0, \dots, f_n$ form a basis of $L(D)$.

$\Rightarrow f_0', \dots, f_m' \in L(D)$ are linear combinations of $f_0, \dots, f_n$.

W.l.o.g. (after invertible linear transformations on $\mathbb{P}^n, \mathbb{P}^m$ and omitting zeroes), we can assume that $m \leq n$, $f_0' = f_0, \dots, f_m' = f_m$.

If there were a point $P \in C(\bar{K})$ such

$\varphi(P) = [x_0 : \dots : x_n] = \left[\frac{f_0}{t_P^{c_P}}(P) : \dots : \frac{f_n}{t_P^{c_P}}(P)\right]$

satisfies $x_0 = \dots = x_m = 0$, then

$v_P(f_0), \dots, v_P(f_m) > -c_P$.

$\Rightarrow f_0, \dots, f_m \in L(D-P)$, so $f_0, \dots, f_m$ don't actually have associated divisor D! $\lightning$

$\Longrightarrow$ We can apply Thm A.2.8. $\square$

Remark A.3.3 If φ, ψ are defined by functions $f_0, \dots, f_n$ and $g_0, \dots, g_m$ with divisors D, E, then we obtain a morphism η defined by functions $f_0 g_0, f_0 g_1, \dots, f_n g_m$ with associated divisor $D+E$ (since $\min\limits_{i,j} v_p(f_i g_j) = \min\limits_i v_p(f_i) + \min\limits_j v_p(g_j)$)

where $\eta(P) = \varphi(P) \otimes \psi(P)$
$\uparrow$
Segre embedding

and that $h(\eta(P)) \approx h(\varphi(P)) + h(\psi(P))$ $\forall P$.
$\uparrow$
Thm A.2.3

Pf of Thm A.3.1

For any D, choose D_1, D_2 s.t. there $D = D_1 - D_2$ and there are morphisms φ_1, φ_2 corresponding to D_1, D_2. Let $h_D(P) := h(\varphi_1(P)) - h(\varphi_2(P))$.

i) If φ is a morphism assoc. to D, then

$h(\varphi(P)) \underset{\varphi}{\approx} h_D(P)$ because

$h(\underset{\substack{\wr \\ D = D_1 - D_2}}{\varphi}(P)) + h(\underset{\substack{\wr \\ D_2}}{\varphi_2}(P)) \approx h(\underset{\substack{\wr \\ D_1}}{\varphi_1}(P))$ by

Lemma A.3.2 and Remark A.3.3.

ii) Let $\underset{\substack{\geq \\ \varphi_1}}{D_1} - \underset{\substack{\geq \\ \varphi_2}}{D_2}$ ... $D = D_1 - D_2$, $D' = D'_1 - D'_2$,

$$D = \underset{\substack{\geq \\ \varphi_1}}{D_1} - \underset{\substack{\geq \\ \varphi_2}}{D_2}, \quad D' = \underset{\substack{\geq \\ \varphi'_1}}{D'_1} - \underset{\substack{\geq \\ \varphi'_2}}{D'_2},$$

$$D + D' = \underset{\substack{\geq \\ \psi_1}}{E_1} - \underset{\substack{\geq \\ \psi_2}}{E_2} \quad \text{as above.}$$

$$h_{D+D'}(P) = h(\psi_1(P)) - h(\psi_2(P))$$

$$h_D(P) = h(\varphi_1(P)) - h(\varphi_2(P))$$

$$h_{D'}(P) = h(\varphi'_1(P)) - h(\varphi'_2(P))$$

$\Rightarrow h_{D+D'}(P) \approx h_D(P) + h_{D'}(P)$ because

$$\underset{\substack{\geq \\ E_1}}{h(\psi_1(P))} + \underset{\substack{\geq \\ D_2}}{h(\varphi_2(P))} + \underset{\substack{\geq \\ D'_2}}{h(\varphi'_2(P))} \approx \underset{\substack{\geq \\ E_2}}{h(\psi_2(P))} + \underset{\substack{\geq \\ D_1}}{h(\varphi_1(P))} + \underset{\substack{\geq \\ D'_1}}{h(\varphi'_1(P))}$$

by Lemma A.3.2 and Remark A.3.3 (applied 4 times) because $E_1 + D_2 + D'_2 = E_2 + D_1 + D'_1$.

$\square$

Thm A.3.4 If $\psi : C \to C'$ is a nonconstant morphism between smooth proj. curves over K and $h_D : C(\bar{K}) \to \mathbb{R}$ for $D \in \mathrm{Div}(C)$ and $h'_{D'} : C'(\bar{K}) \to \mathbb{R}$ for $D' \in \mathrm{Div}(C')$ are the corresponding height functions, then

$$h_{\psi^*(D')}(P) \underset{D',\psi}{\approx} h'_{D'}(\psi(P)) \qquad \forall P \in C(\bar{K}).$$

Pf We can assume (by ii) that there is a morphism $\varphi : C' \to \mathbb{P}^n$ defined by $f_0, \dots, f_n$ with divisor D'.

$\Rightarrow$ The morphism $\varphi \circ \psi : C \to \mathbb{P}^n$ is defined by $f_0 \circ \psi, \dots, f_n \circ \psi$ with divisor $D = \psi^*(D')$ (because $\mathrm{div}(f_i \circ \psi) = \psi^*(\mathrm{div}(f_i))$.)

$\Rightarrow h_{\psi^*(D')}(P) \approx h(\varphi(\psi(P))) \approx h'_{D'}(\psi(P))$.

$\square$

2.4. Heights of points on elliptic curves

Let E be an elliptic curve over K and let

$$\varphi : E \longrightarrow \mathbb{P}^2_K \quad , \quad \psi : E \longrightarrow \mathbb{P}^1_K$$

be the morphisms defined in section 2.1 (corr. to divisor $3[O]$, $2[O]$).

⇝ We get height function

$$h(\varphi(P)) \approx h_{3[O]}(P) \approx 3\, h_{[O]}(P)$$

$$h(\psi(P)) \approx h_{2[O]}(P) \approx 2\, h_{[O]}(P).$$

<u>Rmk</u> $\varphi(O) = [0:1:0]$

So the projection $[x:y:z] \longmapsto [x:z]$
$\qquad\qquad\qquad\quad \varphi(P) \longmapsto \psi(P)$

is not well-def. at $\varphi(O) = [0:1:0]$ (the polynomials x, z have a common zero on the image $\varphi(E)$) although it <u>can</u> be extended to all of E! ($\psi(O) = [1:0]$)

And the height changes under this projection!

<u>Thm 2.4.1</u> $h_{[O]}(nP) \underset{\substack{n \\ (\text{and } E)}}{\approx} n^2 h_{[O]}(P) \quad \forall P \in E(\bar{k})$

<u>Pf</u> By Thm A.3.4 applied to $[n]: E \to E$,

$$h_{[n]^*([O])}(P) \approx h_{[O]}([n](P)) \quad \forall P \in E(\bar{k}).$$

But $[n]^*([O]) = \sum_{P \in E[n]} [P] = n^2[O] + \sum_{P \in E[n]} ([P]-[O])$

[n] unram.

lies in the same divisor class as $n^2[O]$

because $\sum_{P \in E[n]} P = O$ as $E[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$

and $\sum_{v \in (\mathbb{Z}/n\mathbb{Z})^2} v = \sum_{x,y \in \mathbb{Z}/n\mathbb{Z}} \binom{x}{y} = 0.$

$$\Rightarrow h_{[n]^*([O])}(P) \approx h_{n^2[O]}(P) \approx n^2 h_{[O]}(P).$$

$\square$