Cor For any morphism $\begin{array}{ccc} f: V & \longrightarrow & W \\ \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\ \mathbb{A}^n_k & & \mathbb{A}^m_k \end{array}$ and

any point $P \in V(k)$, the map

$$f^*: \Gamma(W) \longrightarrow \Gamma(V)$$

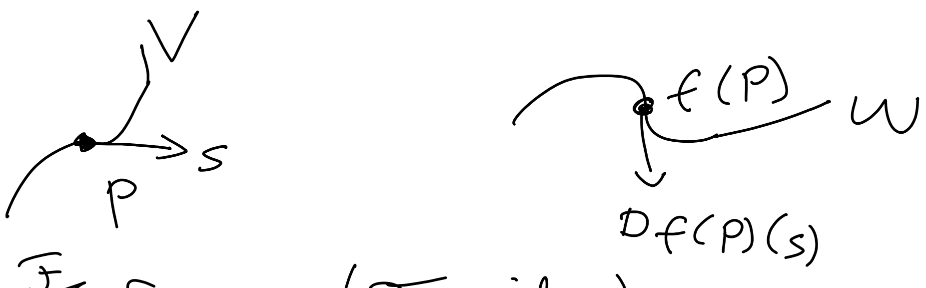
induces a (well-defined!) linear map

$$\begin{array}{ccc} Df^*(P): T^*_{W,f(P)} & \longrightarrow & T^*_{V,P} \\ \parallel & & \parallel \\ m_{W,f(P)}/m^2_{W,f(P)} & & m_{V,P}/m^2_{V,P} \end{array}$$

and therefore a dual map

$$Df(P): T_{V,P} \longrightarrow T_{W,f(P)}$$

(the derivative of f at P).



Rmk For any (Zariski) open neighborhood U of P, if you let $n = m_{V,P}\,\mathcal{O}_V(U)$ be the ideal of $\mathcal{O}_V(U)$ generated by $m_{V,P}$, then $n/n^2 \cong T^*_{V,P}$.

Same with $\mathcal{O}_{V,P}$ instead of $\mathcal{O}_V(U)$!

("Tangent spaces only depend on points close to U").

Rmk This explains why $f: \mathbb{A}^1_k \to V = V(y_1^2 - y_2^3) \subseteq \mathbb{A}^2_k$, $x \mapsto (x^3, x^2)$

isn't an isomorphism:

The derivative $Df(0): \underset{\substack{\| \\ k}}{T_{\mathbb{A}^1,0}} \longrightarrow \underset{\substack{\| \\ k^2}}{T_{V,(0,0)}}$

isn't an isomorphism.

Thm Let $V \subseteq \mathbb{A}^n_k$ be a smooth curve and let $P \in V(k)$. Then, the ring $\mathcal{O}_{V,P}$ is a discrete valuation ring with maximal ideal $m_{V,P}\mathcal{O}_{V,P}$ of functions vanishing at P. We denote the valuation by v_P.

Intuitively, $v_P(f)$ is the <u>multiplicity</u> of the root of f at P.

An element f of $\mathcal{O}_{V,P}$ with $v_P(f) = 1$, is called a <u>uniformiser at P</u>.

Rmk $f \in \mathcal{O}_{V,P}$ is a uniformiser at P if and only if $f(P) = 0$ and $\underbrace{Df(P)}_{\text{map } T_{V,P} \to T_{\mathbb{A}^1,0}} \neq 0$.

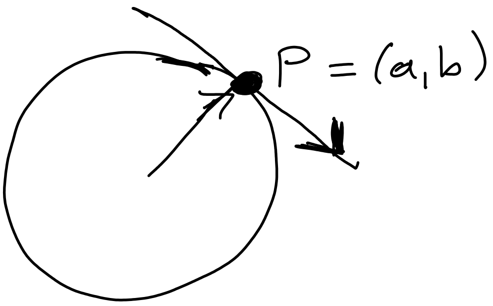
$T_{V,P} \cong k$, $T_{\mathbb{A}^1,0} \cong k$

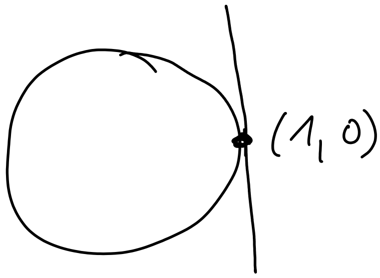
Ese $V = V(x^2+y^2-1) \subseteq \mathbb{A}^1_k$

$P = (a, b) \in V \qquad a^2 + b^2 = 1$

$f = X - a$ is a uniformizer at P

if and only if $b \neq 0$.





$T_{V,P} = \langle (b, -a) \rangle$

If $P = (1,0)$, then $V_P(X-1) = 2$:

$Y - 0$ is a uniformizer at P (same reason as before) and

$$\frac{X-1}{Y^2} = \frac{X-1}{1-X^2} = -\frac{1}{X+1},$$

$\boxed{X^2+Y^2-1=0}$

which has value $-\frac{1}{2} \neq 0$ at $P = (1,0)$.

$$\Rightarrow V_P\left(\frac{X-1}{Y^2}\right) = 0$$

$$V_P(X-1) - 2V_P(Y) = V_P(X-1) - 2.$$

$xb - ya$ is always a uniformiser at any point $P = (a,b)$.

1.6. Differentials

Def Let A be a k-algebra. Its module of differentials is the quotient $\Omega_k(A) = F/Q$, where F is the free A-module with basis $([x])_{x\in A}$ and $Q \subseteq F$ is the submodule generated by elements of the following forms:

a) $[x+y] - [x] - [y]$ with $x, y \in A$

b) $[\lambda x] - \lambda [x]$ with $\lambda \in k$, $x \in A$

c) $[xy] - x[y] - y[x]$ with $x, y \in A$.

The image of $[x]$ in $\Omega_k(A)$ is called dx.

Rmk a) $d(x+y) = dx + dy$

b) $d(\lambda x) = \lambda dx$

c) $d(xy) = xdy + ydx$

(Define differentiation for polynomials)

Thm $\Omega_k(k[X_1,\dots,X_n])$ is the free $k[X_1,\dots,X_n]$-module with basis $dX_1,\dots,dX_n$.

We have $df = \frac{\partial f}{\partial X_1}\cdot dX_1 + \dots + \frac{\partial f}{\partial X_n}\cdot dX_n$.

Thm Let $A = k[X_1,\dots,X_n]/I$ for any ideal I of $k[X_1,\dots,X_n]$. Then, $\Omega_k(A) = F'/Q'$, where F' is the free A-module with basis $dX_1,\dots,dX_n$ and Q' is the A-module

$$Q' = \{ df \quad | f \in I\}$$

Pf HW $\square$

Rmk If $I = (f_1,\dots,f_m)$, then Q' is generated by $df_1,\dots,df_m$.

Ex $\Omega_k(k[X,Y]/(X^2+Y^2-1))$

$=$ (free mod. with basis (dX, dY)) / (module gen. by $2X\,dX + 2Y\,dY$).

Def Let $V \subseteq \mathbb{A}^n_K$, $I = I(V)$

and $r \in \Omega_K(\underbrace{\Gamma(V)}_{K[X_1,\dots,X_n]/I})$
$\quad\; \|$
$g_1(X)\,dX_1 + \dots + g_n(X)\,dX_n$

$(g_i \in K[X_1,\dots,X_n]/I)$.

To any point $P \in V(K)$, we can then associate the element

$$\underbrace{g_1(P)}_{\in K}\,dX_1 + \dots + \underbrace{g_n(P)}_{\in K}\,dX_n$$

of the cotangent space $T^*_{V,P}$

(where we identify dX_i with the map $K^n \longrightarrow K$ as before).
$(y_1,\dots,y_n) \longmapsto y_i$

Def Let $V \subseteq \mathbb{A}^n_K$ be irreducible. For any open $U \subseteq V$, let $\Gamma(U, \Omega_V) = \Omega_V(U)$
$:= \Omega_K(\mathcal{O}_V(U))$

("set of rational differentials defined at every point in U") (Ω_K = "cotangent bundle")

__Rmk__ $\Omega_V(U) = \Omega_V(V) \otimes_{\mathcal{O}_V(V)} \mathcal{O}_V(U)$.

__Rmk__ For any morphism $f : \underset{\mathbb{A}^n_K}{\overset{}{\underset{\cap}{V}}} \longrightarrow \underset{\mathbb{A}^m_K}{\overset{}{\underset{\cap}{W}}}$

and any open $U \subseteq W$, we obtain

a map $Df^* : \Omega_W(U) \longrightarrow \Omega_V(\underbrace{f^{-1}(U)}_{\subseteq V \text{ open}})$.

1.7. Projective varieties

Def The n-dimensional projective space $\mathbb{P}^n_K$ over K is the set of lines through the origin in $(n+1)$-dimensional affine space $\overline{K}^{n+1}$. The line through $(x_0, \dots, x_n) \neq 0$ by $[x_0 : \dots : x_n] \in \mathbb{P}^n_K$.

Rmk $\mathbb{P}^n_K$ is covered by $n+1$ subsets $H_0, \dots, H_n$, where

$$H_i = \{ [x_0 : \dots : x_n] \in \mathbb{P}^n_K \mid x_i \neq 0 \}.$$

For any i, we get a bijection

$$\varphi_i : \quad H_i \longrightarrow \mathbb{A}^n_K$$

$$[x_0 : \dots : x_n] \longmapsto \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i} \right)$$

omit!

$$[x_0^{(i)} : \dots : x_{i-1}^{(i)} : \underset{x_i^{(i)}}{\overset{}{1}} : x_{i+1}^{(i)} : \dots : x_n^{(i)}] \longleftarrow (x_0^{(i)}, \dots, \widehat{x_i^{(i)}}, \dots, x_n^{(i)})$$

Def A projective variety defined over K is a subset $V \subseteq \mathbb{P}^n_K = \{[x_0 : \dots : x_n] \mid x_0, \dots, x_n \in \bar{K}\}$ such that $\varphi_i(V \cap H_i)$ is a subvariety of $\mathbb{A}^n_K$ defined over K for all i.

We then write $V \subseteq \mathbb{P}^n_K$.

For $V \subseteq \mathbb{P}^n_K$, we write $V(K) = V \cap \mathbb{P}^n_K(K)$ where $\mathbb{P}^n_K(K) = \{[x_0 : \dots : x_n] \mid x_0, \dots, x_n \in K\}$.

(So $V(\bar{K}) = V$.)

The closed subsets of $\mathbb{P}^n_K$ w.r.t. the Zariski topology (over K) are the proj. varieties $V \subseteq \mathbb{P}^n_K$.

Rmk The topology on $\mathbb{A}^n_K \overset{\varphi_i}{\cong} H_i \subseteq \mathbb{P}^n_K$ is the subspace top.

Rmk The topology on $\mathbb{P}^n_K$ is obtained by "glueing" the topologies on $H_0, \dots, H_n$.

Def To a homogeneous polynomial $f \in K[X_0, \ldots, X_n]$, we associate the set $V(f) = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n_K(\overline{K}) \mid f(x_0, \ldots, x_n) = 0\}$ $\subseteq \mathbb{P}^n_K$

independent of the choice of representative $(x_0, \ldots, x_n)$ of $[x_0 : \ldots : x_n]$ because f is homogeneous

Rmk $V(f)$ is a projective variety with $\varphi_i(V(f) \cap H_i) = V(f_i) \subseteq \mathbb{A}^n_K$, where

$$f_i = f(x_0^{(i)}, \ldots, x_{i-1}^{(i)}, 1, x_{i+1}^{(i)}, \ldots, x_n^{(i)}) \in K[\underbrace{x_0^{(i)}, \ldots, x_{i-1}^{(i)}, x_{i+1}^{(i)}, \ldots, x_n^{(i)}}_{n \text{ variables}}]$$

Ese The hyperplane $\overline{H_i} := V(X_i = 0) \subseteq \mathbb{P}^n_K$ is a proj. var. $\Rightarrow H_i = \mathbb{P}^n_K \setminus \overline{H_i}$ is an open subset of $\mathbb{P}^n_K$.

Def If $I \subseteq k[X_0, \ldots, X_n]$ is an ideal generated by homogeneous polynomials, we write

$$V(I) = \bigcap_{\substack{f \in I \\ \text{homogeneous}}} V(f) \subseteq \mathbb{P}^n_k .$$

Thm Every $V \subseteq \mathbb{P}^n_k$ is of the form $V = V(I)$ for I as above.