Rmk $\emptyset \neq U \subseteq U' \subseteq X \implies \mathcal{O}_X(U) \supseteq \mathcal{O}_X(U')$.

Rmk $$K(X) = \bigcup_{\substack{\emptyset \neq U \subseteq X \\ \text{open}}} \mathcal{O}_X(U)$$

Rmk $\mathcal{O}_X$ is called the sheaf of functions on X.

Rmk For $U = X$, this agrees with the previous definition $\Gamma(X) = \mathcal{O}_X(X) = k[x_1, \dots, x_n]/I(X)$.

Def For an irreducible closed $Y \subseteq X$ (e.g. a point in $X(k)$), the ring of rational functions on X defined around Y is
$$\mathcal{O}_{X,Y} := \{ t \in K(X) \mid t \text{ defined at } P \ \forall P \in Y(\bar{k}) \}.$$

Rmk $Y \subseteq Y' \subseteq X \implies \mathcal{O}_{X,Y} \supseteq \mathcal{O}_{X,Y'}$

Rmk $$\mathcal{O}_{X,Y} = \bigcup_{\substack{Y \subseteq U \subseteq X \\ \text{open}}} \mathcal{O}_X(U)$$

1.3. Dimension

<u>Def</u> The <u>dimension</u> of an irreducible affine variety V is the transcendence degree of the field $K(V)$.

<u>Rmk</u> $V(\overline{K})$ is a finite set if and only if $\dim(V) = 0$.

<u>Def</u>
- V is a <u>curve</u> if $\dim(V) = 1$.
- <u>surface</u> if $\dim(V) = 2$.

<u>Ex</u> $V = V(x^2+1) \subset \mathbb{A}^1_{\mathbb{R}}$ irred.

$V(\mathbb{R}) = \emptyset, \quad V(\mathbb{C}) = \{\pm i\}$.

$\Gamma(V) = \mathcal{O}_V(V) = \mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$

$K(V) = \mathbb{C}$ alg. extension of $\mathbb{R}$

$\Rightarrow \dim(V) = \text{transcendence degree} = 0$.

Ex $V = V(0) \subset \mathbb{A}^1_k$ irred.

$\Gamma(V) = \mathcal{O}_V(V) = k[x]$

$k(V) = k(x)$

$\Rightarrow \dim(V) =$ transcendence degree $= 1$.

$U = V \setminus \{a_1, \dots, a_r\}$ $(a_1, \dots, a_r \in k)$

$$\Rightarrow \mathcal{O}_V(U) = \left\{ \frac{f}{(x-a_1)^{e_1} \cdots (x-a_r)^{e_r}} \,\middle|\, f \in k[x],\ e_1, \dots, e_r \geq 0 \right\}.$$

$Z = \{a\}$ $(a \in k)$

$$\Rightarrow \mathcal{O}_{V,Z} = \left\{ \frac{f}{g} \,\middle|\, f, g \in k[x],\ g(a) \neq 0 \right\}.$$

Ex $V = V(x^2 + y^2 - 1) \subset \mathbb{A}^2_k$ irred.

$\mathcal{O}_V(V) = k[x,y]/(x^2+y^2-1)$

$k(V) =$ quotient field of $\mathcal{O}_V(V)$

$= k(x)[y]/(x^2+y^2-1)$

$\Rightarrow \dim(V) =$ tr. deg $= 1$.

1.4. Morphisms

From now on, assume that the field K is perfect (every alg. ext. of K is separable), e.g. $\mathrm{char}(K)=0$.

Def Let $V \subseteq \mathbb{A}^n_K$ and $W \subseteq \mathbb{A}^m_K$.

A morphism $f: V \to W$ is a map $f: V(\overline{K}) \longrightarrow W(\overline{K})$ which is given by regular functions: There exist $f_1, \dots, f_m \in \Gamma(V)$ such that

$$f(x) = (f_1(x), \dots, f_m(x)) \in \mathbb{A}^m_K \quad \forall x \in V(\overline{K}).$$

Rmk We obtain a K-algebra hom.

$$\begin{array}{ccc} f^*: \Gamma(W) & \longrightarrow & \Gamma(V) \\ \| & & \| \\ K[Y_1, \dots, Y_m]/I(W) & & K[X_1, \dots, X_n]/I(V) \\ Y_i & \longmapsto & f_i \end{array}$$

sending $a \in \Gamma(W)$ to $a \circ f \in \Gamma(V)$.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, "a \circ f = f^*(a)"'] & W \arrow[d, "a"] \\ & \mathbb{A}^1_K \end{tikzcd}$$

Rmk We get a bijection

$$\{\text{morphism } f: V \to W\} \longleftrightarrow \{K\text{-alg. hom. } \Gamma(W) \to \Gamma(V)\}.$$

Exe If $W(\overline{K})$ consists of just one point, then $W(K) = W(\overline{K})$.

There is exactly one morphism $V \to W$ for any V.

$$\Gamma(W) = K[X_1, \dots, X_n]/(X_1 - a_1, \dots, X_n - a_n) = K$$

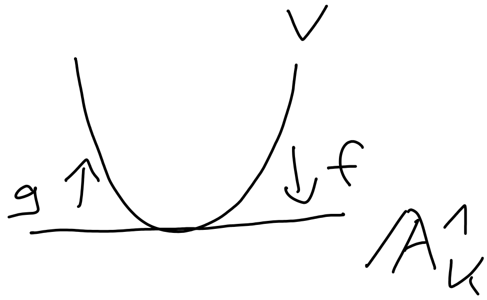
if $W(K) = \{a_1, \dots, a_n\}$.

Exe We have a bij.

$$\{\text{morphism } f: V \to \mathbb{A}^n_K\} \longleftrightarrow \{(f_1, \dots, f_m) \mid f_i \in \Gamma(V)\}.$$

__Ex__ $f: \mathbb{A}^n_K \to \mathbb{A}^m_K$ any linear map.

__Ex__ Let $V = V(y_2 - y_1^2) \subset \mathbb{A}^2_K$

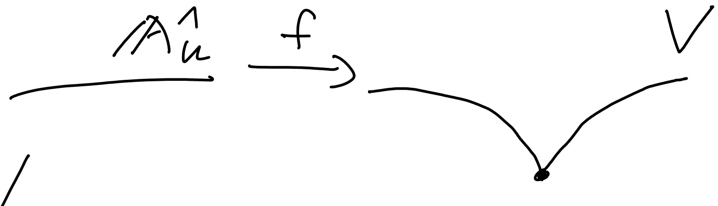


$$f: V \to \mathbb{A}^1_K$$
$$(y_1, y_2) \mapsto y_1$$

is an isomorphism with inverse

$$g: \mathbb{A}^1_K \to V$$
$$x \mapsto (x, x^2)$$

__Ex__ Let $V = V(y_1^2 - y_3^3) \subset \mathbb{A}^2_K$



$$f: \mathbb{A}^1_K \longrightarrow V$$
$$x \longmapsto (x^3, x^2)$$

is a morphism.
The map $f: \mathbb{A}^1_K(L) \longrightarrow V(L)$

$$\mathbb{A}^1_K(L) = L, \qquad V(L) = \{(y_1,y_2) \in L^2 \mid y_1^2 = y_2^3\}$$

is a bijection for any field $K \subseteq L \subseteq \overline{K}$.

But f is <u>not</u> an isomorphism:
The k-alg. hom.

$$f^*: k[Y_1,Y_2]/(Y_1^2-Y_2^3) \longrightarrow k[X]$$
$$Y_1 \mapsto X^3$$
$$Y_2 \mapsto X^2$$

isn't surjective! (The image doesn't contain X.)

This has to do with the tangent space of V at $(0,0)$.

1.5. Tangent spaces

Def The tangent space to $V = V(I) \subset \mathbb{A}^n_K$ at a point $P \in V(K)$ is the K-vector space

$$T_{V,P} := \bigcap_{f \in I} \ker(Df(P)) \subseteq K^n$$

where $Df(P) : K^n \longrightarrow K$ is the Jacobian map of f at P. ("derivative")

Rmk It suffices to consider generators $f_1, \dots, f_r$ of the ideal I.

Pf (sketch)

Product rule: $Dfg(P) = \underbrace{f(P)}_{0} \cdot Dg(P) + g(P) \cdot Df(P)$

for $f \in I$, $g \in K[x_1, \dots, x_n]$

$\Rightarrow$ If $Df(P)(a) = 0 \Rightarrow Dfg(P)(a) = 0$. $\square$

Def Its dual $T^*_{V,P}$ is the cotangent space.

Rmk $T_{V,P}^{*} \overset{A}{=} (K^n)^* / \{Df(P) \mid f \in I\}$

(as a K-vector space).

Pf Every lin. fct. $T_{V,P} \to K$ is the restriction of a lin. fct. $t : K^n \to K$.

The restriction of t to $T_{V,P}$ is zero if and only if for all $a \in K^n$:

If $Df(P)(a) = 0 \ \forall f \in I$, then $t(a) = 0$.

This is equivalent to

$t \in$ span of $\{Df(P) \mid f \in I\}$

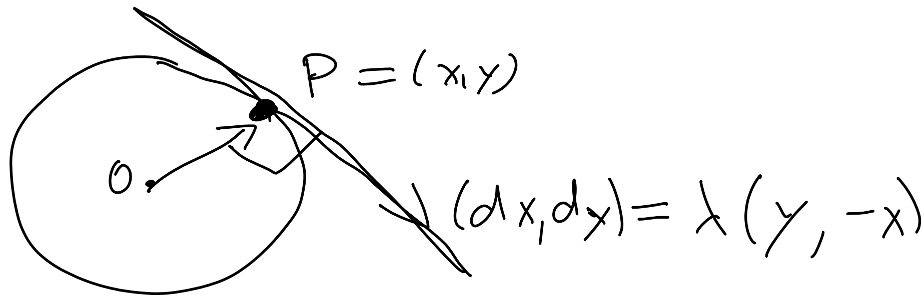
$= \{Df(P) \mid f \in \cancel{I}\}$. $\square$

Ex $V = V(0) = \mathbb{A}^n_K$

$\Rightarrow T_{V,P} = K^n$.

Ex $V = V(\underbrace{x^2+y^2-1}_{f}) \subset \mathbb{A}^2_k$

$P = (x,y) \in V(k) \qquad (x^2+y^2=1)$



$$Df(P) : k^2 \longrightarrow k$$

$$(dx,dy) \longmapsto 2x\,dx + 2y\,dy$$

$$\Rightarrow T_{V,P} = \ker(Df(P))$$

$$= \{(dx,dy) \in k^2 \mid x\,dx + y\,dy = 0\}$$

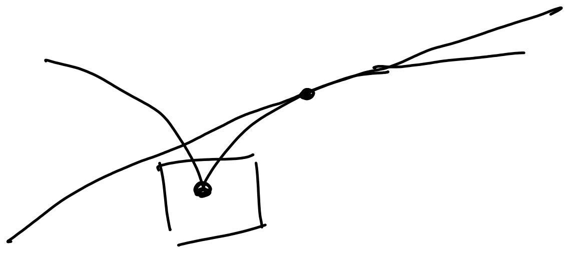
$$= \{(a,b) \in k^2 \mid xa + yb = 0\}$$

$$\underset{\uparrow \; x \text{ or } y \neq 0}{=} \langle (y,-x) \rangle_k$$

$$T^*_{V,P} = \langle \underbrace{dx, dy}_{\text{standard basis of } (k^2)^*} \rangle / \langle x\,dx + y\,dy \rangle$$

Ex $V = V(x^2 - y^3) \subset \mathbb{A}^2_K$

$P = (x, y) \in V(K)$ $\qquad (x^2 = y^3)$



$Df(P): K^2 \longrightarrow K$

$(dx, dy) \longmapsto 2x\,dx - 3y^2\,dy$

$$\Rightarrow T_{V,P} = \begin{cases} \langle (3y^2, 2x) \rangle, & x, \cancel{y} \neq 0 \\ K^2, & x = y = 0 \end{cases}$$

Rmk If $V \subseteq \mathbb{A}^n_K$ is irreducible, then $\dim(T_{V,P}) \geq \dim(V)$ $\forall P \in V(\overline{K})$.

Def An irreducible variety $V \subseteq \mathbb{A}^n_K$ is smooth if $\dim(T_{V,P}) = \dim(V)$ $\forall P \in V(\overline{K})$.

Otherwise, V is singular. The points P with $\dim(T_{V,P}) > \dim(V)$ are the singular points of V.

Ex $V(X^2+Y-1) \subset \mathbb{A}^2_K$ is smooth

Ex $(0,0)$ is the singular point of

$$V(X^2-Y^3) \subset \mathbb{A}^2_K.$$

Def We denote the vanishing ideal of a point $P=(a_1,\dots,a_n) \in K^n$ by

$$m_P = (X_1-a_1,\dots,X_n-a_n).$$

If $V \subseteq \mathbb{A}^n_K$ and $P \in V(K)$, so $m_P \supseteq I(V)$, we let $m_{V,P} \subseteq \Gamma(V)$ be the image of m_P in $\Gamma(V) = K(X_1,\dots,X_n)/I(V)$.
(It's an ideal of $\Gamma(V)$.)

Thm There's a natural isomorphism

$$m_{V,P}/m^2_{V,P} \cong T^*_{V,P}$$

of K-vector spaces.

Pf W.l.o.g. $P = (0, \dots, 0)$.

$m_P = (X_1, \dots, X_n)$.

$$m_{V,P} / m_{V,P}^2 \cong m_P / (I(V) + m_P^2)$$

$$= (X_1, \dots, X_n) / (I(V) + (X_1^2, X_1X_2, \dots, X_n^2))$$

We have $f(P) = 0 \quad \forall f \in I(V)$.

Consider

$$J = \left\{ \sum_i \frac{\partial f}{\partial X_i}(P) \cdot X_i \;\middle|\; f \in I(V) \right\}$$

the set of linearizations of elements of $I(V)$.
(k-vector space)

$$\Rightarrow m_{V,P} / m_{V,P}^2 \cong (X_1, \dots, X_n) / ((J) + (X_1^2, X_1X_2, \dots))$$

$$= \langle X_1, \dots, X_n \rangle / J$$

$$= T_{V,P}^* .$$

Remk A