Math 137: Algebraic Geometry

Spring 2021

Problem set #7

due Friday, March 26 at noon

Problem 1. Let $V_1, \dots, V_m \subseteq K^n$ be nonempty algebraic subsets. Show that

$$\dim(V_1 \cup \cdots \cup V_m) = \max(\dim(V_1), \dots, \dim(V_m)).$$

Problem 2 (Compare Problem 1 on problem set 1)**.** Let $P_1, \dots, P_m \in K^n$ (with $m \geqslant 1$). Show that there are n polynomials $f_1, \dots, f_n \in K[X_1, \dots, X_n]$ such that $\{P_1, \dots, P_m\} = V(f_1, \dots, f_n)$. (Hint: Use a projection and induction over n.)

Problem 3.

a) Compute the join of the lines

$$V = \{(x, y, z) \in K^3 \mid y = z = 0\}$$

and

$$W = \{(x, y, z) \in K^3 \mid x = 0,\ z = 1\}$$

in K^3. Is it Zariski dense in K^3?

b) (double bonus) Show that there are one-dimensional irreducible algebraic subsets $V, W \subseteq \mathbb{C}^3$ whose join is $\mathbb{C}^3$.

Problem 4. Let $n, d \geqslant 1$, let $S \subseteq K^n$ be a finite set and let T be the vector space of polynomials $f \in K[X_1, \dots, X_n]$ of degree at most d such that $S \subseteq V(f)$.

a) Show that if $|S| \geqslant 1$, then

$$\dim(T) \leqslant \binom{n+d}{n} - 1.$$

b) Show that if $|S| \geqslant 2$, then

$$\dim(T) \leqslant \binom{n+d}{n} - 2.$$

Problem 5. Let $n \geqslant 2$ and $d \geqslant 1$. Consider the vector space $F_d \cong K^{\binom{n+d}{n}}$ of polynomials f in $K[X_1, \ldots, X_n]$ of degree at most d. Show that there is a function $0 \neq r \in \Gamma(F_d)$ (a polynomial in the $\binom{n+d}{n}$ coefficients of f) such that $r(f) = 0$ for all reducible polynomials $f \in F_d$.

Problem 6 (bonus)**.** Consider the algebraic set

$$W = \{(x, y, z) \in K^3 \mid x^2 + y^2 + z^2 = 1\},$$

the (dominant finite) projection $\varphi : W \to K^2$ sending (x, y, z) to (x, y), and the subset

$$V = \{(x, y, z) \in W \mid z^3 + x + 1 = 0\}.$$

Find a polynomial $g \in K[X, Y]$ such that $\varphi(V) = \{(x, y) \in K^2 \mid g(x, y) = 0\}$.