# Math 137: Algebraic Geometry

## Spring 2021

### Problem set #6

due Friday, March 19 at noon

**Problem 1.** Which of the following morphisms are finite? (Say for $K = \mathbb{C}$.)

a) The morphism $K^2 \to K$ sending $(x, y)$ to $x^3y + xy^3 + 3x + 1$.

b) The morphism $K \to K^2$ sending $x$ to $(x^2, x^3)$.

**Problem 2.** Let $\varphi : V \to W$ be a morphism. Show that if $V$ is the union of algebraic subsets $V_1, \ldots, V_n$ and each restriction $\varphi : V_i \to W$ is a finite morphism, then $\varphi$ is a finite morphism.

**Problem 3.** Let $V \subseteq K^n$ be a finite set and let $W$ be any algebraic set. Show that every map $\varphi : V \to W$ is a finite morphism.

**Problem 4.** Let $\varphi : V \to W$ be a dominant morphism between irreducible algebraic sets. Assume $\Gamma(V)$ is generated by $n$ elements as a $\varphi^*(\Gamma(W))$-module. Show that the preimage of any point $Q \in W$ has size at most $n$.

**Problem 5.** Let $A \subseteq K^n$ and $B \subseteq K^m$ be irreducible algebraic subsets.

a) Show that the algebraic subset $A \times B \subseteq K^{n+m}$ is irreducible.

b) Show that $\dim(A \times B) = \dim(A) + \dim(B)$. (Hint: At least for "$\geqslant$", use Noether normalization.)

c) (bonus) Show that the ring $\Gamma(A \times B)$ is isomorphic to the tensor product $\Gamma(A) \otimes_K \Gamma(B)$.

**Problem 6** (bonus)**.** Say $K = \mathbb{C}$. Construct a surjective but nonfinite morphism $\varphi : V \to W$ between irreducible algebraic sets such that every $P \in W$ has only finitely many preimages. (You get half the points if $V$ is reducible.)