Math 137: Algebraic Geometry

Spring 2021

Problem set #5

due Friday, March 12 at noon

Problem 1. Show that no two of the algebraic sets

$$\{(x,y) \in \mathbb{C}^2 \mid x^2 + y^2 = 1\}$$

and

$$\{(x,y) \in \mathbb{C}^2 \mid xy = 0\}$$

and

$$\{(x,y,z) \in \mathbb{C}^3 \mid x^2 + y^3 + z^4 = 1\}$$

and

$$\{(x,y,z) \in \mathbb{C}^3 \mid x + y = 2x + y + z = 0\}$$

are isomorphic.

Problem 2. Are $a = X^2 \in \mathbb{C}(X)$ and $b = X^3 + X + 1 \in \mathbb{C}(X)$ algebraically independent over $\mathbb{C}$? If not, find a polynomial $f \in \mathbb{C}[S,T]$ with $f(a,b) = 0$.

Problem 3. Let $a_1, \ldots, a_n$ be a transcendence basis of a field extension M of L and let $b_1, \ldots, b_m$ be a transcendence basis of a field extension L of K. Show that $a_1, \ldots, a_n, b_1, \ldots, b_m$ is a transcendence basis of the field extension M of K.

Problem 4. Let $\varphi : A \dashrightarrow B$ and $\psi : B \dashrightarrow A$ be dominant rational maps such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identity rational maps on B and A, respectively. Show that there is an open subset $\varnothing \neq U \subseteq A$ and an open subset $\varnothing \neq V \subseteq B$ such that φ is defined at every point in U and ψ is defined at every point in B and such that their restrictions to U and V are bijections $U \to V$ and $V \to U$.

Problem 5. Let $A \subseteq K^n$ and $B \subseteq K^m$ be irreducible algebraic subsets and let $\varphi : A \dashrightarrow K^m$ be a rational map given by rational functions $f_1, \ldots, f_m \in K(A)$. Let $\varnothing \neq U \subseteq A$ be any open subset of A (with respect to the subspace topology) on which $f_1, \ldots, f_m$ are defined. Show that $\varphi(U) \subseteq B$ if and only if $g(f_1, \ldots, f_m) = 0$ in $K(A)$ for every $g \in I(B) \subseteq K[Y_1, \ldots, Y_m]$. (In particular, note that this doesn't depend on the open set U!)