Math 137: Algebraic Geometry

Spring 2021

Problem set #4

due Friday, March 5 at noon

Problem 1. Show that every monomial order $\leqslant$ on $\mathcal{S}(X_1, \dots, X_n)$ is a well-order.

Problem 2. Let G be a Gröbner basis of $I \subseteq K[X_1, \dots, X_n]$. Show that the set $V(I)$ is finite if and only if for all $1 \leqslant i \leqslant n$, there is an element $g \in G$ such that $\mathrm{lm}(g) = X_i^t$ for some $t \geqslant 0$.

Problem 3 (Combinatorial Nullstellensatz)**.** Let $A_1, \dots, A_n$ be finite subsets of K and consider their cartesian product $A = A_1 \times \cdots \times A_n \subset K^n$. Show that $\{f_1, \dots, f_n\}$ is a Gröbner basis of the vanishing ideal $I(A) \subseteq K[X_1, \dots, X_n]$ with respect to every monomial order, where

$$f_i = \prod_{a \in A_i} (X_i - a).$$

Problem 4. Let $A = V(I) \subseteq K^n$ and $B = V(J) \subseteq K^m$ be algebraic subsets and let $\varphi : A \to B$ be a morphism with corresponding ring homomorphism $\varphi^* : \Gamma(B) \to \Gamma(A)$. Show that the Zariski closure of the image $\varphi(A)$ is $V(\ker(\varphi^*)) \subseteq B$.

Problem 5 (bonus)**.** Let $\leqslant$ be a monomial order on $\mathcal{S}(X_1, \dots, X_n, Y_1, \dots, Y_m)$ such that $A < B$ whenever A is contained in $\mathcal{S}(X_1, \dots, X_n)$ but B isn't contained in $\mathcal{S}(X_1, \dots, X_n)$.

Let G be a Gröbner basis of an ideal $I \subseteq K[X_1, \dots, X_n, Y_1, \dots, Y_m]$.

a) Show that $G \cap K[X_1, \dots, X_n]$ is a Gröbner basis of the ideal $I' = I \cap K[X_1, \dots, X_n]$ of $K[X_1, \dots, X_n]$.

b) Show that the Zariski closure of the image of $V(I) \subseteq K^{n+m}$ under the projection to K^n (projecting to the first n coordinates) is $V(I') \subseteq K^n$.

Problem 6 (bonus)**.** Let $\leqslant$ be any monomial order on $K[X_1, \dots, X_n]$ and let $0 \neq f, g \in K[X_1, \dots, X_n]$. Show that if $\gcd(\mathrm{lm}(f), \mathrm{lm}(g)) = 1$, then 0 is a reduction of

$$S(f,g) = \frac{M}{\mathrm{lt}(f)} \cdot f - \frac{M}{\mathrm{lt}(g)} \cdot g$$

with respect to $\{f, g\}$, where $M = \operatorname{lcm}(\operatorname{lm}(f), \operatorname{lm}(g)) = \operatorname{lm}(fg)$.