Math 137: Algebraic Geometry

Spring 2021

Problem set #3

due Friday, February 26 at noon

Problem 1. Let $K = \mathbb{C}$ and for any integers $a, b \geqslant 1$, consider the algebraic subset $V_{a,b} = V(X^b - Y^a)$ of $\mathbb{C}^2$ and the morphism $\varphi_{a,b} : \mathbb{C} \to V_{a,b}$ sending t to (t^a, t^b).

a) For which pairs (a, b) is $\varphi_{a,b}$ injective?

b) For which pairs (a, b) is $\varphi_{a,b}$ surjective?

c) For which pairs (a, b) is $\varphi_{a,b}$ an isomorphism?

d) (bonus) For which pairs (a, b) is $V_{a,b}$ isomorphic to K?

Problem 2. Let $\varphi : A \to B$ be a morphism, where A is an algebraic subset of K^n and B is an algebraic subset of K^m. Show that if A is irreducible, then the (Zariski) closure of the image $\varphi(A) \subseteq K^m$ is irreducible.

Problem 3. a) Consider the algebraic set

$$V = \{(x, y, z) \in K^3 \mid x^2 + y^2 = z^2\}.$$

Find a nonconstant morphism $\varphi : K \to V$. (Hint: Pythagorean triples.)

b) Consider the algebraic set

$$W = \{(x, y) \in K^2 \mid x^2 + y^2 = 1\}.$$

Assuming that the field K has characteristic zero, show that there is no nonconstant morphism $\psi : K \to W$. (Hint: Pythagorean triples.)

Problem 4. Identify the space $M_n(K)$ of $n \times n$-matrices with entries in K with the vector space K^{n^2} (by sending a matrix A to a vector consisting of its entries). For any $r \leqslant n$, consider the subset $V_r \subseteq M_n(K) = K^{n^2}$ of matrices of rank at most r.

a) Show that V_r is an algebraic subset of K^{n^2}.

b) Show that V_r is an irreducible subset of K^{n^2}.