Math 137: Algebraic Geometry

Spring 2021

Problem set #2

due Friday, February 19 at noon

Problem 1. Show that $X = \{(t, e^t) \mid t \in \mathbb{R}\}$ is not an algebraic subset of $\mathbb{R}^2$.

Problem 2. Let K be any field and let A be any subset of K^n. Show that $V(I(A))$ is the closure of A with respect to the Zariski topology. (This is called the Zariski closure of A.)

Problem 3. Let I be an ideal of a polynomial ring $K[X_1, \ldots, X_n]$ over a field K. Let $J = \sqrt{I}$ be its radical. Show that $J^n \subseteq I$ for some $n \geqslant 1$.

Problem 4. Let K be any field and let A and B be algebraic subsets of K^n. Show that there exists an integer $m \geqslant n$ and an algebraic subset C of K^m such that the image of C under the projection $K^m \to K^n$ sending $(x_1, \ldots, x_m)$ to $(x_1, \ldots, x_n)$ is the set difference $A \backslash B$.

Problem 5. Give an example of an algebraic field extension L of a field K that is not module-finite (i.e. not a finite-dimensional vector space).

Problem 6 (bonus)**.** Show that $\mathbb{C}$ is not an algebraic field extension of $\mathbb{Q}$. (Hint: Show that the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ is countable.)

Problem 7. Show Corollary 2.20: If S is an integral ring extension of R and T is an integral ring extension of S, then T is an integral ring extension of R.