Claim 1: $m_P^{r+s} \subseteq (f,g)\mathcal{O}_{\mathbb{A},P}$ if the tangent cones of $V(f)$ and $V(g)$ at P have intersection $\{0\}$.

Pf Let h be a polynomial which vanishes at every ~~point~~ in $V(f,g)\setminus\{P\}$ but not at P.

(exists by CRT!)

$\Rightarrow h \cdot m_P \subseteq I(V(f,g)) = \sqrt{(f,g)}$

$\Rightarrow (h \cdot m_P)^t \subseteq (f,g)$ for suff. large t.

$\underset{\substack{\uparrow \\ h \in \mathcal{O}_{\mathbb{A},P}^\times}}{\Longrightarrow} m_P^t \subseteq (f,g)\mathcal{O}_{\mathbb{A},P}$ " —

We now use downward induction to show this for all $t \geq r+s$.

Assume $t \geq r+s$ and for all $t' > t$, we have $m_P^{t'} \subseteq (f,g)\mathcal{O}_{\mathbb{A},P}$.

Denote the space of hom. deg. d pol. in $K[X,Y]$ by R_d.

We obtain a map

$$\alpha : R_{t'-r} \times R_{t'-s} \longrightarrow R_{t'}$$
$$(a,b) \longmapsto \mathrm{in}(f)a + \mathrm{in}(g)b$$

Since the tangent cones have intersection $\{0\}$, the pol. $\mathrm{in}(f)$, $\mathrm{in}(g)$ are relatively prime. Hence, the kernel of α is $\{(\mathrm{in}(g)q, -\mathrm{in}(f)q) \mid q \in R_{t'-r-s}\}$

$\cong R_{t'-r-s}$.

But $\dim(R_{t'-r}) + \dim(R_{t'-s}) - \dim(R_{t'-r-s})$

$= (t'-r+1) + (t'-s+1) - (t'-r-s+1)$

$= t'+1 = \dim(R_{t'})$ for $t' \geq r+s$.

Hence, α is surjective for $t' \geq r+s$.

$\Rightarrow$ Any $q \in m_p^t$ can be written as

$q = \mathrm{in}(f)a + \mathrm{in}(g)b$ with $a, b \in K[X,Y]$

with $m_p(a) \geq t-r$, $m_p(b) \geq t-s$.

$$\Rightarrow g = \underbrace{fa + gb}_{\in (f,g)\mathcal{O}_{\mathbb{A},P}} - \underbrace{(f - \mathrm{in}(f))a - (g - \mathrm{in}(g))b}_{m_P(\cdot) \geq t+1}$$

$$\in (f,g)\,\mathcal{O}_{\mathbb{A}^2,P}.$$

induction

$\square$ (Claim 1)

$\square$ (Thm)

4.6. Bézout's theorem

We define intersection numbers in $\mathbb{P}^2$ by looking at affine patches.

<u>Def</u> Let $f, g \in K[X,Y,Z]$ be homogeneous. Let $\varphi : K^2 \xrightarrow{\sim} U \subset \mathbb{P}^2_K$ be a chart map arising from an identification $\psi : K^2 \xrightarrow{\sim} T \subset K^3$.

Let $\tilde{f} = f \circ \psi$, $\tilde{g} = g \circ \psi$

(so $\varphi^{-1}(V_{\mathbb{P}^2}(f)) = V_{K^2}(\tilde{f})$,

$\varphi^{-1}(V_{\mathbb{P}^2}(g)) = V_{K^2}(\tilde{g})$).

The intersection number of f, g at $P \in U$ is $I_P(f,g) := I_{\varphi^{-1}(P)}(\tilde{f}, \tilde{g})$.

Thm 4.6.1 The intersection number $I_P(f,g)$ depends only on f, g, P, not on φ, ψ.

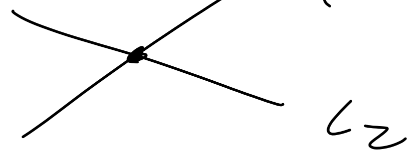
Rmk The intersection number is invariant under projective transformations.

Thm 4.6.2 (Bézout)

Let $f, g \in K[X,Y,Z]$ be hom. of degrees m, n. If $V_{\mathbb{P}^2}(f,g)$ is a finite set, then

$$\underbrace{\sum_{P \in \mathbb{P}^2_K} I_P(f,g)}_{\text{"number of points in } V_{\mathbb{P}^2}(f) \cap V_{\mathbb{P}^2}(g) \text{ with multiplicity"}} = m \cdot n.$$

Exe Any two lines $L_1 \neq L_2$ in $\mathbb{P}^2_K$ intersect in exactly one point, transversally ($m = n = 1$).



__Ex__ Let $g = Z$. $\rightsquigarrow V_{\mathbb{P}^2}(g)$ is a line in $\mathbb{P}^2_K$.
$$V_{\mathbb{P}^2}(g) \cong \mathbb{P}^1_K$$

$f(X,Y,0) \in K[X,Y]$ is hom. of deg. m.

Unless $f(X,Y,0) \neq 0$, this pol. has exactly m roots (with mult) in $\mathbb{P}^1 = V_{\mathbb{P}^2}(g)$.

__Ex__ Assume $\mathrm{char}(K) \neq 2$.

$f = X^2 + (Y-Z)^2 - Z^2 = X^2 + Y^2 - 2YZ$

$g = X^2 + (Y+Z)^2 - Z^2 = X^2 + Y^2 + 2YZ$

$V_{\mathbb{P}^2}(f,g) = \{[0:0:1], [1:i:0], [1:-i:0]\}$

$I_{[0:0:1]}(f,g) = 2$

$I_{[1:\pm i:0]}(f,g) = 1$

__Cor 4.6.3__ If $\# V_{\mathbb{P}^2}(f,g) > mn$ for hom f,g of degrees m,n, then $\# V_{\mathbb{P}^2}(f,g) = \infty$.

__Cor 4.6.4__ Let $S \subset \mathbb{P}^2_K$ be a finite set of points which don't all lie on the same line. If $\# S$ is a prime number, then there are no curves $V_{\mathbb{P}^2}(f)$, $V_{\mathbb{P}^2}(g)$ that intersect exactly in the points of S with intersection number 1 at each point in S.

<u>Pf of Thm 4.6.2</u> Assume f, g are nonconstant.

Consider the standard affine chart

$$\varphi : K^2 \xrightarrow{\sim} U \subset \mathbb{P}^2 .$$
$$(x,y) \longmapsto [x:y:1]$$

W.l.o.g. $V_{\mathbb{P}^2}(f,g) \subseteq U$, so all common roots $[x:y:z]$ of f, g have $z \neq 0$.

$\Rightarrow$ f, g aren't divisible by Z

$\Rightarrow \hat{f}(X,Y) = f(X,Y,1) \in K[X,Y]$,

$\hat{g}(X,Y) = g(X,Y,1) \in K[X,Y]$

have degrees m, n.

$$\Rightarrow \sum_{P \in \mathbb{P}^2} I_P(f,g) = \sum_{P \in K^2} I_P(\hat{f},\hat{g})$$

$$\underset{\text{Thm 4.4.6}}{=} \dim_K\left(K[X,Y]/(\hat{f},\hat{g})\right).$$

Let $R = k[X,Y,Z]$, $R_d = \{h \in R \text{ hom. of deg. } d\}$

$$\Gamma = R/(f,g), \quad \Gamma_d = R_d / (R_d \cap (f,g)).$$

<u>Claim 1</u> $\dim(\Gamma_d) = mn$ for $d \geq m+n$.

<u>Pf</u> Consider the following maps:

$$R_{d-m-n} \xhookrightarrow{\alpha} R_{d-m} \times R_{d-n} \xrightarrow{\beta} R_d \xrightarrow{\pi} \Gamma_d$$

$h \mapsto (gh, -fh)$

$(a,b) \longmapsto fa + gb$

$r \mapsto r \bmod (f,g) \cap R_d$

$\ker(\pi) = \operatorname{im}(\beta)$, $\ker(\beta) = \operatorname{im}(\alpha)$

$$\Rightarrow \dim(\Gamma_d) = \dim(R_d) - \dim(R_{d-m}) - \dim(R_{d-n}) + \dim(R_{d-m-n})$$

$$\underset{d \geq m+n}{=} \binom{d+2}{2} - \binom{d-m+2}{2} - \binom{d-n+2}{2} + \binom{d-m-n+2}{2}$$

$= mn.$

$\square$ (Claim 1)

Claim 2 The map $\Gamma_d \longrightarrow \Gamma_{d+1}$

$r \longmapsto r \cdot z$

is injective for all $d \geq 0$

(hence an isom., for all $d \geq m+n$).

Pf Let $r \cdot z = fa + gb$ with

$a \in R_{d+1-m}$, $b \in R_{d+1-n}$.

$\Rightarrow 0 = f(x,y,0)\,a(x,y,0) + g(x,y,0)\,b(x,y,0)$

Since f, g have no common zeros $(x:y:z)$ with $z=0$, the pol. $f(x,y,0), g(x,y,0)$ must be relatively prime.

$\Rightarrow$ We have $a(x,y,0) = g(x,y,0)\,h(x,y)$

$b(x,y,0) = -f(x,y,0)\,h(x,y)$

for some $h \in k[x,y]$.

$\Rightarrow$ $a \equiv g \cdot h \mod z$

$b \equiv -fh \mod z$

$$\Rightarrow r = \frac{fa+gb}{z} = \frac{f(a-gh) + g(b+fh)}{z}$$

$$= f \cdot \underbrace{\frac{a-gh}{z}}_{\in k(x,y,z)} + g \cdot \underbrace{\frac{b+fh}{z}}_{\in k(x,y,z)} \in (f,g).$$

$\Rightarrow r = 0$ in Γ_d. $\square$

(Claim 2)

<u>Claim 3</u> The map $\Gamma_d \longrightarrow K[X,Y]/(\tilde{f},\tilde{g})$

$r \longmapsto r(X,Y,1)$

is a vector space isomorphism for $d \geq m+n$.

<u>Pf</u> <u>inj</u>: Let $r(X,Y,1) = f(X,Y,1)\tilde{a}(X,Y) + g(X,Y,1)\tilde{b}(X,Y)$

with $\tilde{a}, \tilde{b} \in K[X,Y]$.

Let $a, b \in K[X,Y,Z]$ be homogenisations so that $\tilde{a}(X,Y) = a(X,Y,1)$, $\tilde{b}(X,Y) = b(X,Y,1)$.

Then, $Z^i r = Z^j fa + Z^k gb$ for some $i,j,k \geq 0$ (to make both sides hom. of the same degree).

$\Rightarrow Z^i \cdot r = 0$ in Γ_{d+i}.

$\underset{\text{Claim 2}}{\Rightarrow} r = 0$ in Γ_d.

surj.: Let $\tilde{s} \in K[X,Y]$ and let $s \in K[X,Y,Z]$ be homogeneous of degree $d+t$ $(t \geq 0)$

with $\tilde{s}(X,Y) = s(X,Y,1)$.

Let $r \in \Gamma_d$ be a preimage of s under the isomorphism

$$\Gamma_d \xrightarrow{\sim} \Gamma_{d+t}.$$
$$r \mapsto Z^t \cdot r$$

$\Rightarrow r(X,Y,1) = \tilde{s}$ in $K[X,Y]/(\tilde{f},\tilde{g})$.

$\square$ (Claim 3)

Summary:

$$\sum_{P \in \mathbb{P}^2} I_P(f,g) = \sum_{P \in K^2} I_P(\tilde{f},\tilde{g})$$

$$= \dim\left(K[X,Y]/(\tilde{f},\tilde{g})\right)$$

$$\underset{\text{Claim 3}}{=} \dim(\Gamma_d) \underset{\text{Claim 1}}{=} m \cdot n \text{ for } d \geq m+n.$$

$\square$

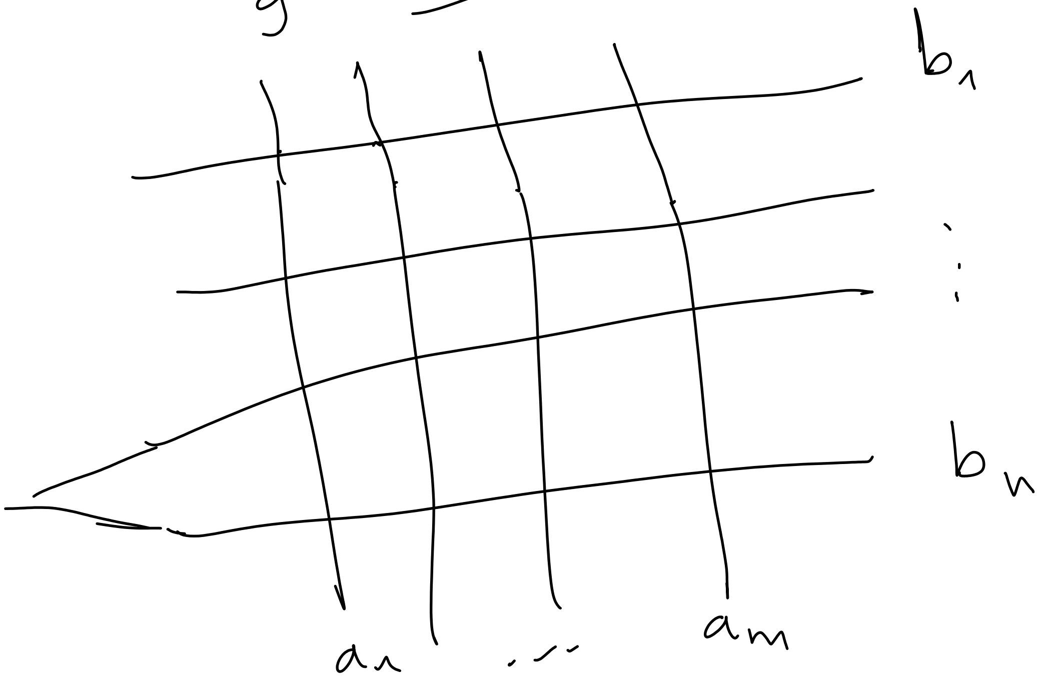
<u>Ese</u> Let $a_1, \dots, a_m$ be lines in $\mathbb{P}^2$
$b_1, \dots, b_n$ — " —

Assume $a_1, \dots, a_m, b_1, \dots, b_n$ are distinct.

Let f be the prod. of the lin. pol. corr. to $a_1, \dots, a_m$.

Let g ... corr. to $b_1, \dots, b_n$.

$\rightsquigarrow$ f hom. of deg. m,
g — " — n.



Then $V_{\mathbb{P}^2}(f) = a_1 \cup \dots \cup a_m$
$V_{\mathbb{P}^2}(g) = b_1 \cup \dots \cup b_n$,

$\Rightarrow V_{\mathbb{P}^2}(f, g) = V_{\mathbb{P}^2}(f) \cap V_{\mathbb{P}}(g)$ consists of at most mn points.

Es. $f = (\text{lin. pol})^m$, $g = (\text{lin. pol.})^n$.