W.l.o.g., $h(P) \neq 0$ for $h = \gcd(f_1 f_2, g)$
(Otherwise, both sides are ∞.)

Consider the following maps

$$0 \to \mathcal{O}_{\mathbb{A}^2,P}/(f_1,g)\ldots \xrightarrow{\alpha} \mathcal{O}_{\mathbb{A}^2,P}/(f_1f_2,g)\ldots \xrightarrow{\beta} \mathcal{O}_{\mathbb{A}^2,P}/(f_2,g)\ldots \to 0$$

$[h] \longmapsto [f_2 h]$
(well-def. because
$f_2(f_1,g) \subseteq (f_1f_2,g)$)

$[h] \longmapsto [h]$
(well-def. because
$(f_1f_2,g) \subseteq (f_2,g)$)

β is surjective: clear

α is injective:

Let $h \in \mathcal{O}_{\mathbb{A}^2,P}$ with $f_2 h \in (f_1 f_2, g)\mathcal{O}_{\mathbb{A}^2,P}$,
so $f_2 h = f_1 f_2 a + g b$ with $a, b \in \mathcal{O}_{\mathbb{A}^2,P}$.
Let $d \in K[X,Y]$ be the least common multiple of the denominators of h, a, b,
so $h = \frac{\hat{h}}{d}$, $a = \frac{\hat{a}}{d}$, $b = \frac{\hat{b}}{d}$ with
$\hat{h}, \hat{a}, \hat{b} \in K[X,Y]$. Since the denom. must be $\neq 0$ at P, we have $d(P) \neq 0$ as well.

$\Rightarrow f_2 \tilde{h} = f_1 f_2 \tilde{a} + g \tilde{b}$

$\Rightarrow f_2 (\hat{h} - f_1 \hat{a}) = g \tilde{b}$

$\Rightarrow g \mid f_2 (\hat{h} - f_1 \hat{a})$ in $K[x,y]$

Let $r = \cancel{\gcd}(f_2, g)$. $\Rightarrow r(P) \neq 0$

$\Rightarrow \frac{g}{r} \mid \hat{h} - f_1 \tilde{a}$ in $K[x,y]$

Write $\hat{h} - f_1 \hat{a} = \frac{g}{r} \cdot s$ with $s \in K[x,y]$.

$$\Rightarrow h = \frac{\hat{h}}{d} = \frac{f_1 \tilde{a} + \frac{g}{r} \cdot s}{d} = f_1 \cdot \frac{\tilde{a}}{d} + g \cdot \frac{s}{rd}$$

$$\in (f_1, g) \mathcal{O}_{\mathbb{A}^2, P}$$

$d(P) \neq 0 \Rightarrow \frac{\tilde{a}}{d} \in \mathcal{O}_{\mathbb{A}^2, P}$

$r(P) \neq 0 \Rightarrow \frac{s}{rd} \in \mathcal{O}_{\mathbb{A}^2, P}$

$\Rightarrow h$ is zero in $\mathcal{O}_{\mathbb{A}^2, P} / (f_1, g) \mathcal{O}_{\mathbb{A}^2, P}$.

<u>$\operatorname{im}(\alpha) = \ker(\beta)$:</u>

"$\subseteq$" $f_2 h \in (f_2, g) \in \mathcal{O}_{\mathbb{A}^2, P}$ $\forall h \in \mathcal{O}_{\mathbb{A}^2, P}$

"$\supseteq$" Let $h \in \mathcal{O}_{\mathbb{A}^2,p}$ with $h \in (f_2, g)\mathcal{O}_{\mathbb{A}^2,p}$,

so $h = f_2 a + g b$ with $a, b \in \mathcal{O}_{\mathbb{A}^2,p}$.

$\Rightarrow h \equiv f_2 a \mod (f_1 f_2, g)\mathcal{O}_{\mathbb{A}^2,p}$,

so $h \in \mathrm{im}(\alpha)$.

Summary:

$$\dim(\mathrm{im}(\beta)) = \dim\left(\mathcal{O}_{\mathbb{A}^2,p} / (f_2, g)\mathcal{O}_{\mathbb{A}^2,p}\right)$$

$$= I_p(f_2, g)$$

$$\dim(\ker(\beta)) = \dim(\mathrm{im}(\alpha))$$

$$= \dim\left(\mathcal{O}_{\mathbb{A}^2,p} / (f_1, g)\mathcal{O}_{\mathbb{A}^2,p}\right)$$

$$= I_p(f_1, g)$$

$$\dim(\text{domain of } \beta) = \dim\left(\mathcal{O}_{\mathbb{A}^2,p} / (f_1 f_2, g)\mathcal{O}_{\mathbb{A}^2,p}\right)$$

$$= I_p(f_1 f_2, g)$$

$$\dim(\text{domain}) = \dim(\mathrm{im}) + \dim(\ker).$$

$\square$

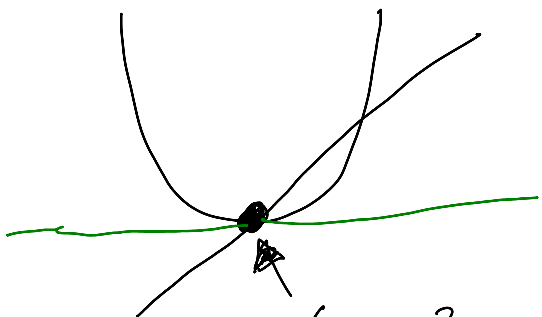
Def Let $0 \neq f, g \in K[x,y]$ be irreducible.
Then, $V(f)$ and $V(g)$ <u>intersect transversally</u> at a point $P \in V(f,g)$ if

$$m_P(f) = m_P(g) = 1$$

and the tangent spaces of $V(f)$ and $V(g)$ at P $\left\{\begin{array}{l}\text{(which are one-}\\ \text{dimensional}\\ \text{vector spaces)}\end{array}\right\}$ have "trivial" intersection $\{0\}$.

Thm 4.5.2 $I_P(f,g) = 1$ if and only if $V(f), V(g)$ intersect transversally at P.

Ex $V(y - x^2)$, $V(y-x)$ intersect transversally at $(0,0)$.



$$I_0(y-x^2, y-x) = I_0(x-x^2, y-x)$$

$I_P(f,g)$ only depends on the ideal (f,g)
$(y-x^2, y-x) = (y-x^2-(y-x), y-x)$

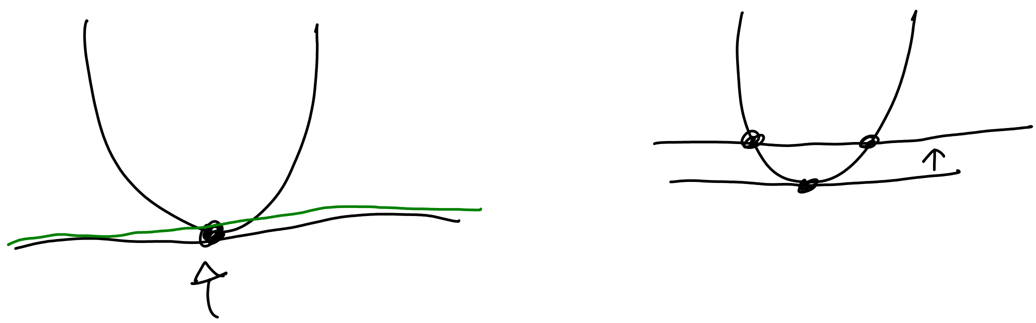
$$= I_0(x(1-x), y-x) = I_0(x, y-x) + I_0(1-x, y-x)$$

$$= I_0(x,y) + 0$$

$(0,0) \notin V(1-x, y-x)$

$$= \dim_k\left(\mathcal{O}_{\mathbb{A}^2,0} / (x,y)\mathcal{O}_{\mathbb{A}^2,k}\right) = 1$$

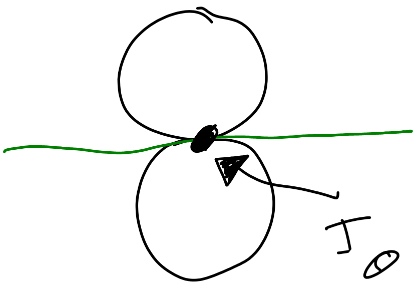
Ex $V(y-x^2)$, $V(y)$ don't intersect transversally at $(0,0)$



$$I_0(y-x^2, y) = I_0(x^2, y)$$
$$= 2\, I_0(x,y) = 2$$

Ex $V(\underbrace{x^2+(y-1)^2-1}_{x^2+y^2-2y})$, $V(\underbrace{x^2+(y+1)^2-1}_{x^2+y^2+2y})$

don't ---



$$I_0(\dots) = I_0(x^2+y^2-2y,\ y)$$
$$= I_0(x^2, y) = 2 I_0(x,y) = 2.$$

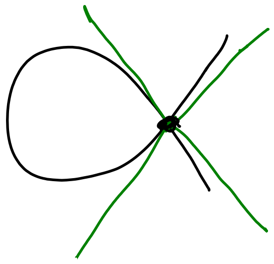
More generally:

Thm 4.5.3 Assume that $h(P) \neq 0$ for $h = \gcd(f,g)$.

Then:

a) $I_p(f,g) \geq m_p(f) \cdot m_p(g)$

b) $I_p(f,g) = m_p(f) \cdot m_p(g)$ if and only if the tangent cones of $V(f)$ and $V(g)$ have "trivial" intersection (no tangent lines in common).



Pf W.l.o.g. $P = (0,0)$.

Let $r = m_p(f)$, $s = m_p(g)$

and $m_p = (X,Y)$ the max. ideal corr. to P.

Consider the following maps:

$$\begin{tikzcd}
K[X,Y]/m_p^s \times K[X,Y]/m_p^r \arrow[r, "\alpha"] & K[X,Y]/m_p^{r+s} \arrow[r, "\beta"] & K[X,Y]/(m_p^{r+s}+(f,g)) \arrow[d, "\sim"', "\text{by pf of Thm 4.4.7 because } V(m_p^{r+s}+(f,g))=\{P\}"] \\
& \mathcal{O}_{\mathbb{A}^2,P}/(f,g)\mathcal{O}_{\mathbb{A}^2,K} \arrow[r] & \mathcal{O}_{\mathbb{A}^2,P}/(m_p^{r+s}+(f,g))\mathcal{O}
\end{tikzcd}$$

$(a,b) \longmapsto fa+gb$

We have $\operatorname{im}(\alpha) = \ker(\beta)$

$$I_P(f,g) = \dim\left(\mathcal{O}_{\mathbb{A}^2,P} / (f,g)\mathcal{O}_{\mathbb{A}^2,P} \right)$$

$$\overset{(I)}{\geq} \dim\left(\mathcal{O}_{\mathbb{A}^2,P} / (m_P^{r+s} + (f,g))\mathcal{O}_{\mathbb{A}^2,P} \right)$$

$$= \dim\left(k[X,Y] / (m_P^{r+s} + (f,g)) \right)$$

$$= \dim\left(k[X,Y] / m_P^{r+s} \right) - \dim(\ker(\beta))$$

$$= \quad \text{''} \quad - \dim(\operatorname{im}(\alpha))$$

$$\overset{(II)}{\geq} \quad \text{''} \quad - \dim(k[X,Y]/m_P^s) - \dim(k[X,Y]/m_P^r)$$

$$= \underbrace{\binom{r+s+1}{2}}_{\substack{\#\text{ mon. in } X,Y \\ \text{of deg.} < r+s}} - \underbrace{\binom{s+1}{2}}_{\substack{\#\text{ mon.} \\ \text{of deg} < s}} - \underbrace{\binom{r+1}{2}}_{\substack{\#\text{ mon.} \\ \text{of deg.} < r}}$$

$$= \frac{(r+s+1)(r+s) - (s+1)s - (r+1)r}{2} = rs$$

$$= m_P(f)\, m_P(g).$$ This shows a).

For b):

Equality in (I) $\Leftrightarrow$ $(f,g)\mathcal{O}_{\mathbb{A}^2,P} = (m_P^{r+s} + (f,g))\mathcal{O}_{\mathbb{A}^2,P}$

$\Leftrightarrow$ $m_P^{r+s} \subseteq (f,g)\mathcal{O}_{\mathbb{A}^2,P}$

Equality in (II) $\Leftrightarrow$ $\alpha : K[X,Y]/m_P^s \times K[X,Y]/m_P^r \to K[X,Y]/m_P^{r+s}$
$(a,b) \mapsto fa+gb$

is injective

Claim 2: α is injective if and only if the tangent cones of $V(f)$ and $V(g)$ have trivial intersection $\{0\}$

Pf "$\Rightarrow$" Assume that they have nontriv intersection.

$\Rightarrow$ The hom. pol. in(f), in(g) of degrees r, s have a linear factor L in common.

$$\Rightarrow \alpha\left(\frac{in(g)}{L}, -\frac{in(f)}{L}\right) = f\cdot\frac{in(g)}{L} - g\cdot\frac{in(f)}{L}$$

$$\equiv f\cdot\frac{g}{L} - g\cdot\frac{f}{L} + \underbrace{f\cdot\frac{in(g)-g}{L} - g\cdot\frac{in(f)-f}{L}}_{\text{only has mon. of deg.} \geq r+s}$$

$$\equiv 0 \mod m_P^{r+s}$$

On the other hand,

$\frac{in(g)}{L}$ has mon. of deg. $s-1$,

so $\frac{in(g)}{L} \neq 0$ in $k(x,y)/m_p^s$.

$\Rightarrow \alpha$ is not injective.

"$\Leftarrow$" If they have triv. intersection, then $in(f)$, $in(g)$ have no common factors.

Let $(a,b) \in \ker(\alpha)$, so

$fa+gb \in m_p^{r+s}$.

If $a \notin m_p^s$, $b \notin m_p^r$, then the lowest degree parts of fa and gb have degrees $< r+s$.

Hence, they need to cancel in $fa+gb$ (since $fa+gb \in m_p^{r+s}$).

$$\Rightarrow \underset{\substack{\| \\ in(f)\,in(a)}}{in(fa)} = \underset{\substack{\| \\ -in(g)\,in(b)}}{-in(gb)}$$

Because $in(f)$, $in(g)$ are relatively prime,
this implies that $in(f) \mid in(b)$
and $in(g) \mid in(a)$, so
in particular $m_p(b) \geq m_p(f) = r$
and $m_p(a) \geq m_p(g) = s$.

$\lightning$ because $a \notin m_p^s$, $b \notin m_p^r$. $\square$

(Claim 2)