Lemma 4.4.3 Let $I \subseteq k[X_1,\dots,X_n]$ and $P \in k^n$.

Then, $m_P(I) = 0 \iff P \notin V(I)$.

Pf $P \notin V(I) \iff \exists\, f \in I : f(P) \neq 0$

$$m_P(I) = 0 \iff \mathcal{O}_{\mathbb{A}^n,P} = I\mathcal{O}_{\mathbb{A}^n,P} \iff \frac{f}{f} = 1 \in I\mathcal{O}_{\mathbb{A}^n,P}$$

$\square$

Rmk If $I \supseteq J$ (and therefore $V(I) \subseteq V(J)$),

then $m_P(I) \leq m_P(J)$.

Pf We have a quotient map

$$\mathcal{O}_{\mathbb{A}^n,P}/J\mathcal{O}_{\mathbb{A}^n,P} \twoheadrightarrow \mathcal{O}_{\mathbb{A}^n,P}/I\mathcal{O}_{\mathbb{A}^n,P},$$

which is surjective. $\square$

Lemma 4.4.4 If $V \subseteq \mathbb{A}^n_k$ is an irreducible algebraic set

with $I = I(V)$, then

$m_P(I) = \dim_k(\mathcal{O}_{V,P})$ for all $P \in V$.

Pf We have an isomorphism

$$\mathcal{O}_{\mathbb{A}^n,P}/I\mathcal{O}_{\mathbb{A}^n,P} \xrightarrow{\sim} \mathcal{O}_{V,P}$$

$$\left[\frac{a}{b}\right] \longmapsto \frac{a}{b} \quad \text{for } a, b \in k[X_1,\dots,X_n],\ b(P) \neq 0.$$

well-def.: $b(P) \neq 0 \Rightarrow b$ is not zero everywhere on V

$a \in I \Rightarrow a$ is zero everywhere on V

$\Rightarrow \frac{a}{b}$ is the zero fct. on V

injective: If $\frac{a}{b}$ is zero on V, then a is zero everywhere on V. $\Rightarrow a \in I$.

surjective: clear. $\square$

Ex $V = \{P\} \subseteq K^n \Rightarrow m_P(I(V)) = 1$.

Pf 1 is a basis of $\mathcal{O}_{V,P}$.
Any rat. fct. on V is given by its value at P. $\square$

Cor 4.4.5 If I is any ideal and W is an irred. comp. of $V(I)$ of dimension ≥ 1, then $m_P(I) = \infty$.

Ex $I = (0) \subset K[X]$, $W = K$, $P = 0$

$\mathcal{O}_{\mathbb{A}^1,P} = \{ \frac{a}{b} \mid b(0) \neq 0 \}$

is ∞-dimensional:

$1, X, X^2, \ldots$ are linearly independent.

Pf of Cor $W \subseteq V(I)$

$$\Rightarrow I(W) \supseteq I(V(I)) = \sqrt{I} \supseteq I$$

$$\Rightarrow m_p(I) \geq m_p(I(W)).$$

$\leadsto$ w.l.o.g. $I = I(W)$.

$$\underset{\text{Lemma}}{\Rightarrow} \quad m_p(I) = \dim_k(\mathcal{O}_{W,p})$$

$$\underset{\mathcal{O}_{W,p} \supseteq \Gamma(W)}{\geq} \dim_k(\Gamma(W))$$

$$\underset{\substack{\text{Lemma } 2.34, \\ \text{Cor } 2.35}}{=} \# W = \infty.$$

$\square$

Thm 4.4.6 Let $I \subseteq K[X_1, \dots, X_n]$ be any ideal. Then,

$$\sum_{P \in V(I)} m_P(I) = \dim_K(K[X_1, \dots, X_n]/I).$$

Proof By Lemma 2.34, Cor. 2.35, if I is a radical ideal, then

$$\# V(I) = \dim_K(\dots).$$

Ex $I = (f)$, $0 \neq f \in K[X]$, $\deg(f) = d$.

$\Rightarrow 1, X, \dots, X^{d-1}$ form a basis of $K[X]/I$, so $\dim(\dots) = d$.

And we have d roots with multiplicities.

Cor 4.4.7 If I is a radical ideal with $\# V(I) < \infty$, then $m_P(I) = 1$ $\forall P \in V(I)$.

Lemma 4.4.8 Let $P \in K^n$ with maximal ideal $m_P = V(\{P\}) \subset K[X_1, \dots, X_n]$. Then, any $f \in K[X_1, \dots, X_n]$ with $f \notin m_P$ is invertible in $\underbrace{K[X_1, \dots, X_n]/m_P^t}_{R_t :=}$ for all $t \geq 0$.

<u>Analogy</u> Any number $n \notin 3\mathbb{Z}$ is invertible in $\mathbb{Z}/3^t\mathbb{Z}$ for all $t \geq 0$.

<u>Pf</u> The mult. by f map $\alpha: R_t \longrightarrow R_t$
$$g \longmapsto fg$$

is K-linear.

R_t is a finite-dimensional K-vector spaces. (If $P=0$, the monomials of degree $< t$ form a basis of R_t.)

If the map α is not an isomorphism, it's not injective, so there is some $g \in K[x_1, \dots, x_n]$ with

$g \notin m_p^t$ but $fg \in m_p^t$

$\Downarrow$ $\qquad\qquad\qquad$ $\Downarrow$

$m_p(g) < t$ $\qquad\qquad$ $m_p(fg) \geq t$

$$m_p(fg) = \underbrace{m_p(f)}_{0} + m_p(g)$$

$\lightning$ $\square$

Pf of Thm Let $V(I) = \{P_1, \dots, P_r\}$ and let $m_{P_1}, \dots, m_{P_r}$ be the corr. max. ideals.

Goal: $K[x_1, \dots, x_n]/I \xrightarrow{\alpha} \prod_{i=1}^{r} \mathcal{O}_{\mathbb{A}^n, P_i} / I\mathcal{O}_{\mathbb{A}^n, P_i}$

$f \longmapsto (f, \dots, f)$

$\sqrt{I} = I(V(I)) = I(\{P_1, \dots, P_r\}) = m_{P_1} \cap \dots \cap m_{P_r}$.

Let $I \supseteq (m_{P_1} \cap \dots \cap m_{P_r})^d$ with $d \geq 1$.

Since the sets $V(m_{P_1}), \dots, V(m_{P_r})$ are pairwise disjoint, the Chinese remainder theorem tells us that

$$(m_{P_1} \cap \dots \cap m_{P_r})^d = (m_{P_1} \cdots m_{P_r})^d$$
$$= m_{P_1}^d \cdots m_{P_r}^d$$
$$= m_{P_1}^d \cap \dots \cap m_{P_r}^d$$

and

$$K[x_1, \dots, x_n]/(m_{P_1} \cap \dots \cap m_{P_r})^d \cong \prod_{i=1}^{r} K[x_1, \dots, x_n]/m_{P_i}^d .$$

In particular, there are polynomials $e_1, \dots, e_n \in K[X_1, \dots, X_n]$ such that

$e_i \equiv 1 \mod m_{p_i}^d$ and $e_i \equiv 0 \mod m_{p_j}^d$ for all $j \neq i$.

<u>α is injective:</u>

Let $f \in K(X_1, \dots, X_n)$,

$f = \frac{a_i}{b_i}$ where $a_i \in I$, $b_i(p_i) \neq 0$ for $i = 1, \dots, n$.

$b_i \notin m_{p_i} \Rightarrow b_i$ is invertible $\mod m_{p_i}^d$, so

there is a polynomial $t_i \in K(X_1, \dots, X_n)$ with $t_i b_i \equiv 1 \mod m_{p_i}^d$

and $t_i \equiv 0 \mod m_{p_j}^d$ for $j \neq i$.

$\Rightarrow t_i b_i = e_i$

$$\Rightarrow f \equiv \underbrace{\Big(\sum_i e_i\Big)}_{1 \mod (m_{p_1} \cap \dots \cap m_{p_r})^d \;\Rightarrow 1 \mod I} f = \sum_i t_i b_i f = \sum_i \underbrace{t_i}_{\in K(X_1, \dots, X_n)} \underbrace{a_i}_{\in I} \in I$$

α surjective Let $\frac{a_i}{b_i} \in \mathcal{O}_{\mathbb{A}^n, P_i}$, $b(P_i) \neq 0$.

Take t_i as before.

Let $f := \sum_i t_i a_i \mod m_{P_i}^d$ for all i

$\Rightarrow f \equiv \sum t_i a_i \mod I \mathcal{O}_{\mathbb{A}^n, P_i}$ for all i.

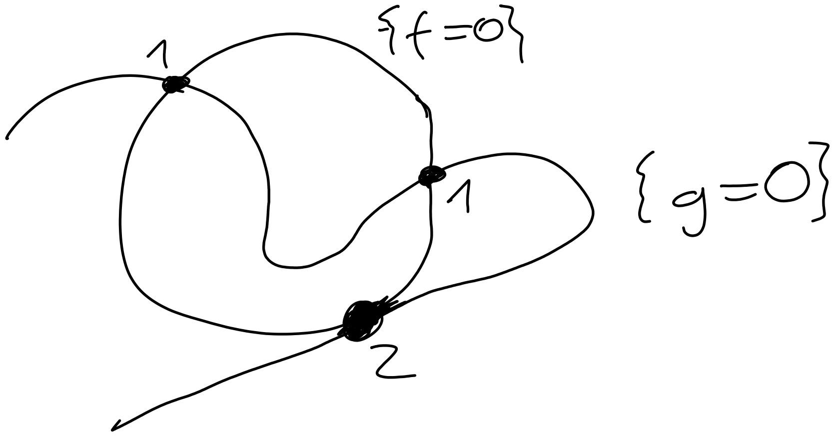
$\square$

4.5. Intersection numbers

Def The intersection number of

$f \in K[X, Y]$ and $g \in K[X, Y]$ at $P \in K^2$ is

$$I_P(f, g) := m_P((f, g))$$
$$= \mathcal{O}_{\mathbb{A}^2, P} / (f, g)\mathcal{O}_{\mathbb{A}^2, P} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}.$$



Lemma 4.5.1

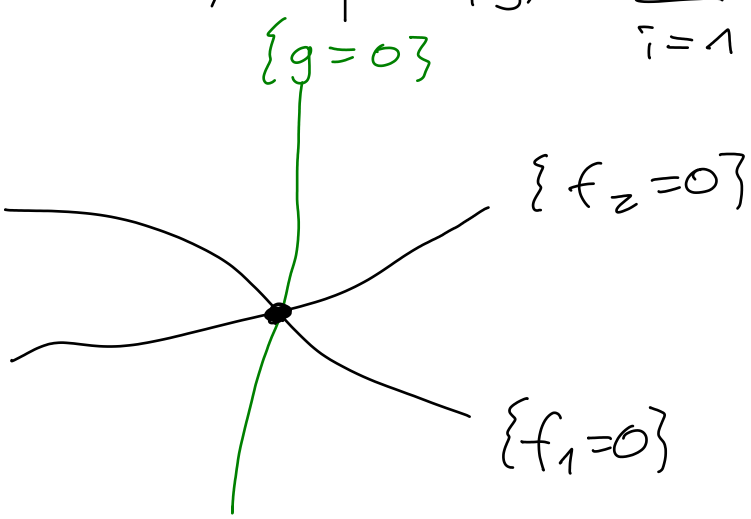
a) $I_P(f,g) = 0 \iff P \notin V(f,g)$

b) $I_P(f,g) = \infty \iff P$ is contained in an irred. comp. of $V(f,g)$ of dimension ≥ 1

$\iff h(P) = 0$ for $h = \gcd(f,g)$.

Lemma 4.5.3 Let $f = f_1 \cdots f_n$ and $g = g_1 \cdots g_m$.

Then, $I_P(f,g) = \sum_{i=1}^{n} \sum_{j=1}^{m} I_P(f_i, g_j)$.



Pf By induction, it suffices to show that $I_P(f_1 f_2, g) = I_P(f_1, g) + I_P(f_2, g)$.