Def The tangent cone of an alg. set V at O is the tangent cone of $I(V)$ at O.

The tangent cone of V at $P \in K^n$ is the tangent cone of the translate $V-P$ at O.

Lemma 4.2.1 For any ideal I, the tangent cones of I and $\sqrt{I}$ at P agree.

Pf $f \in \sqrt{I} \iff \exists m \geq 1 : f^m \in I$

$$\overset{\uparrow}{\underset{\text{in}(f^m) = \text{in}(f)^m}{\Longrightarrow}} \quad V(\text{in}(I)) = V(\text{in}(\sqrt{I})).$$

$\square$

Lemma 4.2.2 The tangent cone of $A \cup B$ at P is the union of the tangent cones at A and B at P.

Pf HW.

Prop 4.2.3 If V is irred. of dimension d, then the tangent cone of V at any point $P \in V$ has dimension d.

4.3. Tangent spaces

Def Let $V \subseteq K^n$ be an alg. set containing O.
The tangent space $T_O(V)$ to V at O is the vector space

$$V_{K^n}(\{ f_1 \text{ hom. deg. 1 part of } f \in I(V) \}).$$

The tangent space $T_P(V)$ to V at $P \in V$ is the tangent space to $V - P$ at O.

Rmk The tangent space always contains the tangent cone.

Pf The hom. deg. 1 part f_1 of $f \in I(V)$ with $f(P) = 0$ is

$$f_1 = \begin{cases} \mathrm{in}(f) & \text{if } m_P(f) = 1 \\ 0 & \text{if } m_P(f) \geq 2. \end{cases}$$ □

Rmk Let $0 \neq f \in K[x_1, \ldots, x_n]$ be squarefree.
The tangent space to $V(f)$ at $P \in V(f)$ is

$$T_P(V(f)) = \begin{cases} \text{the tangent cone at } P & \text{if } m_P(f) = 1, \\ K^n & \text{if } m_P(f) \geq 2. \end{cases}$$

Rmk In general,

$$V(\{f \text{ hom. deg. } 1 \text{ part of } f \in I\})$$
$$\cup\!\!\!\!\!\!\,\text{ \#}$$
$$V(\{f \text{ hom. deg. } 1 \text{ part of } f \in \sqrt{I}\}).$$

$$\big(\text{E.g. } \underbrace{I = (x^2)}_{V(\ldots) = K}, \quad \underbrace{\sqrt{I} = (x)}_{V(\ldots) = \{0\}}.\big)$$

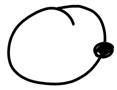
Thm 4.3.1 Let $V \subseteq K^n$ be irreducible.
Then, $\dim_K(T_P(V)) \geq \dim(V)$.

Pf $T_P(V) \supseteq \underbrace{\text{tangent cone}}_{\dim = d \text{ by Prop 4.2.3}}$ $\square$

Def An irred. alg. set $V \subseteq K^n$ is smooth at $P \in V$ if $\dim_K(T_P(V)) = \dim(V)$.

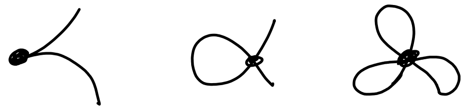
It is smooth if it is smooth at every $P \in V$.
(otherwise singular)

Rmk If V is smooth at P, tangent space = tangent cone.

Ex B  smooth at 0:

$$V = \{(x,y)\in K^2 \mid \underbrace{(x-1)^2+y^2-1}_{x^2+y^2-2x}=0\}$$

$T_0(V) = \{(x,y)\in K^2 \mid x=0\}$ has dim 1.

Ex C, D, E  singular at 0.

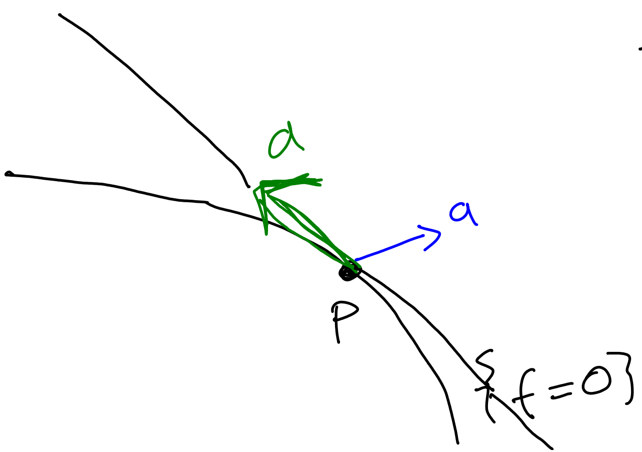
Note: The hom. deg. 1 part of $f\in K[X_1,\dots,X_n]$ is

$$\frac{\partial f}{\partial X_1}(0)\cdot X_1 + \dots + \frac{\partial f}{\partial X_n}(0)\cdot X_n$$

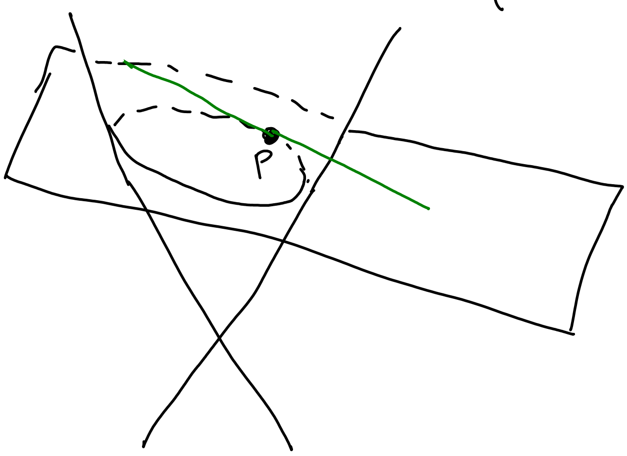
Rmk Hence, $T_P(V) = \bigcap_{f\in I(V)} \ker((Df)(P))$

where $(Df)(P): K^n \to K$ denotes the derivative of $f: K^n \to K$ at P.

" $a \in T_P(V) \iff$ derivative of every $f\in I(V)$ in the direction a is 0 "



$\underline{\text{Eg}}$ $V = V(\underbrace{x^2+y^2-z^2}_{f}, \underbrace{2x+z-11}_{g})$



$P=(x,y,z)$

$(Df)(x,y,z)(a,b,c) = 2xa+2yb-2zc$

$\quad \text{- --} \; (dx,dy,dz) = 2x\,dx + 2y\,dy - 2z\,dz$

$(Dg)(x,y,z) = (a,b,c) = 2a + c$

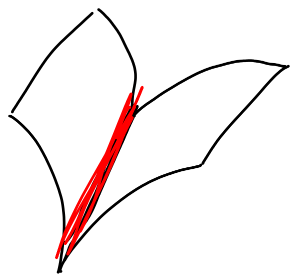
$\quad \text{· - -} \; 2dx + dz$

$$T_{(x,y,z)}(V) = \ker \begin{pmatrix} 2x & 2y & -2z \\ 2 & 0 & 1 \end{pmatrix}$$

$P=(3,4,5) \in V$

$\rightsquigarrow T_p(V) = \left\langle \begin{pmatrix} -4 \\ 13 \\ 8 \end{pmatrix} \right\rangle$.

Lemma 4.3.2 The set $S \subseteq V$ of singular points is algebraic. (so the set of smooth points is an open subset of V.)



Pf Let $I(V) = (f_1, \dots, f_m)$, $P \in V$.

$$T_P(V) = \bigcap_{g \in I(V)} \ker((Dg)(P)).$$

Write $g = h_1 f_1 + \dots + h_m f_m$ with $h_i \in K[X_1, \dots, X_n]$.

$$(Dg)(P)(a) \underset{\text{product rule}}{=} \sum_{i=1}^{m} \Big((Dh_i)(P)(a) \cdot \overbrace{f_i(P)}^{=0 \text{ because } P \in V} + h_i(P) \cdot (Df_i)(P)(a) \Big)$$

$\Rightarrow$ $T_P(V) = \bigcap_{i=1}^{m} \ker((Df_i)(P))$ is the kernel of the $m \times n$-matrix $M(P) = \left(\frac{\partial f_i}{\partial X_j}(P) \right)_{i,j}$

$\Rightarrow$ S is the set of points $P \in V$ for which $\mathrm{rk}(M(P)) \leq n - \dim(V) - 1 =: r$.

Equivalently: The set of $P \in V$ such that all determinants of $\underbrace{(r+1)\times(r+1)\text{-minors of } M(P)}_{\text{pol. in the coord. of } P}$ vanish.

$\square$

Prop 4.3.2 There is a smooth point on any irred. alg. set $V \subseteq K^n$.

Cor 4.3.3 The set of smooth points is a dense open subset of V.

4.4. Multiplicity in finite sets

Def Let I be an ideal of $K[X_1, \dots, X_n]$.

~~Assume $V(I) \subseteq K^n$ is a finite set.~~

The multiplicity of $P \in K^n$ in I is

$$m_P(I) := \dim_K \left(\mathcal{O}_{\mathbb{A}^n_K, P} / I \mathcal{O}_{\mathbb{A}^n_K, P} \right),$$

where

$$\mathcal{O}_{\mathbb{A}^n, P} = \{ f \in \underset{\substack{\| \\ K(X_1, \dots, X_n)}}{K(\mathbb{A}^n)} \text{ def. at } P \}$$

$$= \left\{ \frac{a}{b} \;\middle|\; a, b \in K[X_1, \dots, X_n],\ b(P) \neq 0 \right\}$$

is the local ring of $\mathbb{A}^n = K^n$ at P and

$$I\,\mathcal{O}_{\mathbb{A}^n,p} = \left\{ \frac{a}{b} \;\middle|\; a \in I,\ b \in K[x_1,\dots,x_n],\ b(p) \neq 0 \right\}$$

is the ideal of $\mathcal{O}_{\mathbb{A}^n,p}$ generated by I.

<u>Ex</u> Let $f(X) = \prod_{i=1}^{r} (X - c_i)^{e_i} \in K[X]$

with $c_1,\dots,c_r \in K$ distinct, $I = (f)$.

Let $t \in K$.

$$\mathcal{O}_{\mathbb{A}^1,t} = \left\{ \frac{a}{b} \;\middle|\; a,b \in K[X],\ b(t) \neq 0 \right\}$$

$$\mathcal{O}^{\times}_{\mathbb{A}^1,t} = \left\{ \frac{a}{b} \;\middle|\; a,b \in K[X],\ a(t), b(t) \neq 0 \right\}$$

$$I\,\mathcal{O}_{\mathbb{A}^1,t} = \prod_{i=1}^{r} \underbrace{(X-c_i)^{e_i}}_{\text{unit unless } c_i = t} \cdot \mathcal{O}_{\mathbb{A}^1,t}$$

$$= \begin{cases} \mathcal{O}_{\mathbb{A}^1,t} & \text{if } t \notin \{c_1,\dots,c_r\} \\ (X-c_i)^{e_i}\mathcal{O}_{\mathbb{A}^1,t} & \text{if } t = c_i \end{cases}$$

$$= (X-t)^e\,\mathcal{O}_{\mathbb{A}^1,t} \text{ if } t \text{ is a root of mult. } e \text{ of } f.$$

Lemma 4.4.1 We have a K-algebra isom.

$$\mathcal{O}_{\mathbb{A}^1,t} / (x-t)\mathcal{O}_{\mathbb{A}^1,t} \xrightarrow{\sim} K$$
$$g \longmapsto g(t)$$

Pf well-def: If $g \in (x-t)\mathcal{O}_{\mathbb{A}^1,t}$, then $g(t)=0$.

injective: If $\frac{a(t)}{b(t)} = 0$, then $a(t)=0$, so

$x-t \mid a(t)$ in $K[x]$, so

$$\frac{\frac{a}{x-t}}{b} \in \mathcal{O}_{\mathbb{A}^1,t}.$$

surjective: const. fct. $\in \mathcal{O}_{\mathbb{A}^1,t}$. $\square$

Cor 4.4.2 $\dim_K\left(\mathcal{O}_{\mathbb{A}^1,t}/(x-t)^e\mathcal{O}_{\mathbb{A}^1,t}\right) = e$ for any $e \geq 0$.

Pf Consider the chain of K-vector spaces

$$\mathcal{O}_{\mathbb{A}^1,t} \supseteq (x-t)\mathcal{O}_{\mathbb{A}^1,t} \supseteq (x-t)^2\mathcal{O}_{\mathbb{A}^1,t} \supseteq \dots \supseteq (x-t)^e\mathcal{O}_{\mathbb{A}^1,t}$$

It suffices to prove that

$$\dim_K\left((x-t)^i\mathcal{O}_{\mathbb{A}^1,t}/(x-t)^{i+1}\mathcal{O}_{\mathbb{A}^1,t}\right) = 1 \quad \forall i \geq 0.$$

But we have an isomorphism

$$\mathcal{O}_{\mathbb{A}^1,t}/(x-t)\mathcal{O}_{\mathbb{A}^1,t} \xrightarrow{\sim} (x-t)^i\mathcal{O}_{\mathbb{A}^1,t}/(x-t)^{i+1}\mathcal{O}_{\mathbb{A}^1,t}$$

$$g \longmapsto (x-t)^i g$$

and the LHS has dimension 1 by Lemma 4.4.1. $\square$

<u>Ex</u> (as before) $0 \neq f \in K[X]$, $I = (f)$

$\Rightarrow m_t(I) = e$ if f has a root of mult. e at t (with $e = 0$ if f doesn't have a root at t).

$1, X-t, \dots, (X-t)^{e-1}$ form a basis of $\mathcal{O}_{\mathbb{A}^n,P}/I\mathcal{O}_{\mathbb{A}^n,P}$.