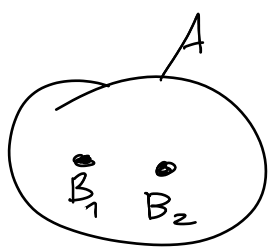
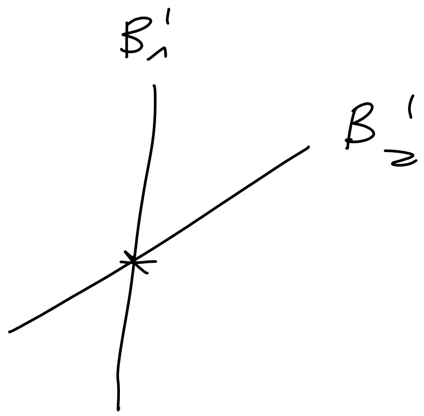
Thm 3.5.3 We can define the **irreducable components** of an alg. set $A \subseteq \mathbb{P}^n_K$ like for $A' \subseteq K^n$ in Thm 2.2.7 and they satisfy the same properties. Furthermore, we have a bij.

$$\{\text{irred. comp. } B \text{ of } A\} \longleftrightarrow \{\text{irred. comp. } B' \neq \{0\} \text{ of } C(A)\}$$

$$B \longleftrightarrow C(B)$$





3.6. Dimension

Def The dimension of an alg. set $\emptyset \neq V \subseteq \mathbb{P}^n_K$ is the largest length d of a chain

$V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_d \subseteq V$ of irred. alg. subsets V_i.

Thm 3.6.1

For any alg. $\emptyset \neq V \subseteq \mathbb{P}^n_K$:

a) $\dim(V) = \dim(C(V)) - 1$

b) If V is irreducible and $\varphi: K^n \longrightarrow \mathbb{P}^n_K$ is an affine patch with $\varphi^{-1}(V) \neq \emptyset$

$$\dim(V) = \dim(\varphi^{-1}(V)).$$

Rmk b) can fail if V is reducible:

e.g. $V = \{\text{point}\} \cup (\text{line at infinity})$

$\Rightarrow \varphi^{-1}(V) = \{\text{point}\}$.

Pf By Thm 3.5.3, we can assume that V is irreducible (even in a)).

For any chain

$$V_0 \subsetneq \cdots \subsetneq V_d \subseteq V \text{ of irred. sets,}$$

we obtain a chain

$$\{0\} = C(\emptyset) \subsetneq C(V_0) \subsetneq \cdots \subsetneq C(V_d) \subseteq C(V) \text{ of irred. sets.}$$

$$\Rightarrow \dim(C(V)) \geq \dim(V) + 1$$

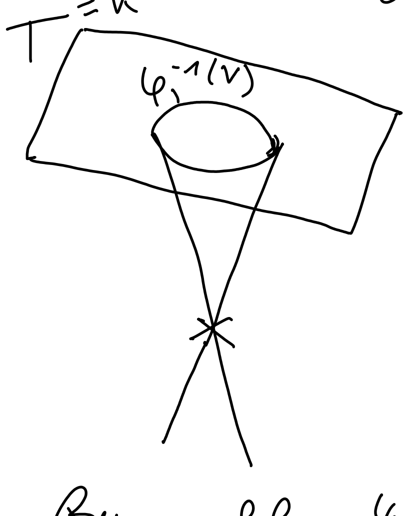
For any chain

$$W_0 \subsetneq \cdots \subsetneq W_d \subseteq \varphi^{-1}(V) \text{ of irred. sets,}$$

we obtain a chain

$$\overline{\varphi(W_0)} \subsetneq \cdots \subsetneq \overline{\varphi(W_d)} \subseteq V \text{ of irred. sets.}$$

$$\Rightarrow \dim(V) \geq \dim(\varphi^{-1}(V)).$$



Let $0 \notin T \subseteq K^{n+1}$ be an n-dim. affine lin. subspace corr. to φ. Then $C(V)$ is the Zariski closure of the join of $\{0\}$ and $C(V) \cap T \cong \varphi^{-1}(V)$.

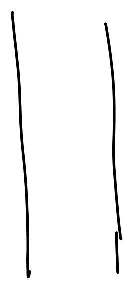
By problem 4 on pset 8, we then have $\dim(C(V)) = \dim(\varphi^{-1}(V)) + 1$. $\square$

Thm 3.6.2 Let $f_1, \dots, f_m \in K[x_0, \dots, x_n]$ be nonconstant hom. pol. with $m \leq n$.
Then, $V_{\mathbb{P}^n_K}(f_1, \dots, f_m) \neq \emptyset$ and every irred. comp. A has $\dim(A) \geq n - m$.

of $V_{\mathbb{P}^n_K}(f_1, \dots, f_m)$

Rmk The first claim is wrong in K^n:

$\emptyset = V(x, x-1) \subseteq K^2$.



Pf By Thm 2.83, every irred. comp. A' of $V_{K^{n+1}}(f_1, \dots, f_m) \subseteq K^{n+1}$ has $\dim(A') \geq n + 1 - m$.

This shows the second claim because any irred. comp. A corr. to an irred. comp. $A' = C(A)$ with $\dim(A) = \dim(A') - 1$.
Since $f_1, \dots, f_m$ are homogeneous of degree ≥ 1, we have $0 \in V_{K^{n+1}}(f_1, \dots, f_m)$.

$\Rightarrow$ There is at least one irred. comp. A'.

It satisfies $\dim(A') \geq n+1-m \geq 1$,

so $A' \neq \{0\}$.

It therefore corresponds to an irred. comp. A of $V_{\mathbb{P}^n_u}(f_1, \dots, f_m)$, so in particular $V_{\mathbb{P}^n_u}(f_1, \dots, f_m) \neq \emptyset$. $\square$

<u>Def</u> If $V, W \subseteq \mathbb{P}^n_u$ irred. alg. sets, the <u>codimension</u> of V in W is

$$\begin{aligned} \operatorname{codim}(V, W) &= \dim(W) - \dim(V) \\ &= \dim(e(W)) - \dim(e(V)) \\ &= \operatorname{codim}(e(V), e(W)). \end{aligned}$$

<u>Thm 3.6.3</u> It's the largest length d of a chain $V = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_d = W$ of irred. alg. sets

<u>Pf</u> HW.

<u>Thm 3.6.4</u> Let $V_1, V_2 \subseteq W \subseteq \mathbb{P}^n_k$ be irred. alg. with $\operatorname{codim}(V_1, W) + \operatorname{codim}(V_2, W) \leq \dim(W)$. Then, $V_1 \cap V_2 \neq \emptyset$ and every irred. comp. A of $V_1 \cap V_2$ satisfies

$$\operatorname{codim}(A, W) \leq \operatorname{codim}(V_1, W) + \operatorname{codim}(V_2, W).$$

<u>Pf</u> Apply Cor 2.8.9 to the affine cones:

Any irred. comp. B of $\ell(V_1) \cap \ell(V_2) = \ell(V_1 \cap V_2)$ satisfies

$$\begin{aligned}\operatorname{codim}(B, \ell(W)) &\leq \operatorname{codim}(\ell(V_1), \ell(W)) \\ &\quad + \operatorname{codim}(\ell(V_2), \ell(W)) \\ &= \operatorname{codim}(V_1, W) + \operatorname{codim}(V_2, W).\end{aligned}$$

This shows the second claim.

For the first claim, use that

$0 \in \ell(V_1) \cap \ell(V_2)$ and

$$\dim(B) \geq \dim(\ell(W)) - \underbrace{(\operatorname{codim}(V_1, W) + \operatorname{codim}(V_2, W))}_{\substack{\leq \dim(W) \\ = \dim(\ell(W)) - 1}}$$

$$\geq 1.$$

$\square$

Exe Any two curves in $\mathbb{P}^2_k$ intersect.

Exe Any curve and surface in $\mathbb{P}^3_k$ intersect.

Exe Any three surfaces in $\mathbb{P}^3_k$ intersect.

4. Multiplicities and tangent spaces

4.1. Multiplicity of a function at a point

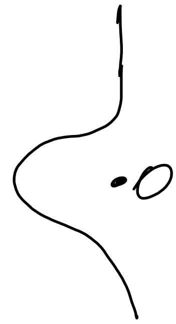
Def Let $0 \neq f \in K[X_1, \ldots, X_n]$.

The multiplicity $m_0(f)$ of f at $P = (0, \ldots, 0)$ is the smallest degree of a monomial occuring in f.

The initial form $in_0(f)$ of f at $P = (0, \ldots, 0)$ is the hom. degree $m_0(f)$ part of f.

Rmk $in_0(f)$ is the lowest order (nonzero) approximation of f at 0.

Ex A $f(x,y) = 7 + x + 2xy^2$

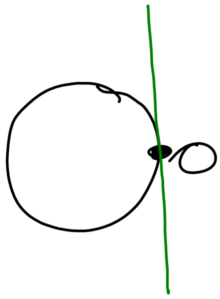


$\rightsquigarrow m_0(f) = 0$, $in_0(f) = 7$

tangent cone $= \emptyset$

Ex B $f(x,y) = (x+1)^2 + y^2 - 1$

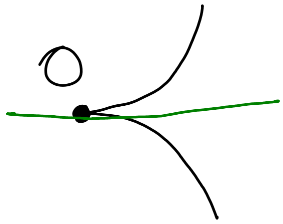
$= x^2 + 2x + y^2$



$\rightsquigarrow m_0(f) = 1$, $in_0(f) = 2x$

$in_0((f)) = (x)$, tangent cone $= \{x = 0\}$

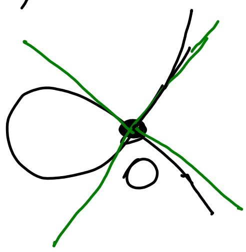
Ex C $f(x,y) = y^2 - x^3$



$\rightsquigarrow m_0(f) = 2$, $in_0(f) = y^2$

$in_0((f)) = (y^2)$, tangent cone $= \{y = 0\}$

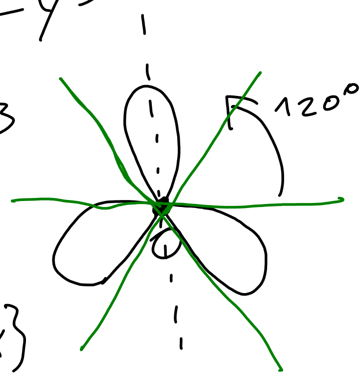
Ex D $f(x,y) = y^2 - x^3 - x^2$



$\rightsquigarrow m_0(f) = 2$, $in_0(f) = y^2 - x^2$

$in_0((f)) = ((y-x)(y+x))$, tangent cone $= \{y = \pm x\}$

Ex E $f(x,y) = (x^2+y^2)^2 + 3x^2y - y^3$



$\rightsquigarrow m_0(f) = 3$, $in_0(f) = 3x^2y - y^3$

$in_0((f)) = (y(\sqrt{3}x - y)(\sqrt{3}x + y))$,

tangent cone $= \{y = 0, \sqrt{3}x, -\sqrt{3}x\}$

Rmk $m_0(f) \geq 1 \iff f(0) = 0$

Rmk $m_0(fg) = m_0(f) + m_0(g)$

$in_0(fg) = in_0(f)\, in_0(g)$

Def For any $P = (a_1, \dots, a_n)$, we let

$m_P(f) = m_0(g)$ for $g(X_1, \dots, X_n) = f(X_1 + a_1, \dots, X_n + a_n) \in K[X_1, \dots, X_n]$

$in_P(f) = in_0(g)$.

Def $P \in K^n$ is a simple root of f if $m_P(f) = 1$.

Rmk $m_P(f)$ is the largest integer $t \geq 0$ such that $f \in \mathfrak{m}_P^t$, where $\mathfrak{m}_P = (X_1 - a_1, \dots, X_n - a_n)$ is the max. id. corresponding to $P = (a_1, \dots, a_n)$.

$$\mathfrak{m}_P^t = ((X_1 - a_1)^t, (X_1 - a_1)^{t-1}(X_2 - a_2), \dots).$$

mon. of deg. t in $X_1 - a_1, \dots, X_n - a_n$

4.2. Tangent cones

Def The initial ideal $in(I)$ of an ideal $I \subseteq K[X_1, \dots, X_n]$ is the homogeneous ideal gen. by the initial forms $in(f)$ of the elements f of I. The tangent cone of I is $V_{K^n}(in(I))$.

(at 0)

Rmk $in_0((f)) = (in_0(f))$.