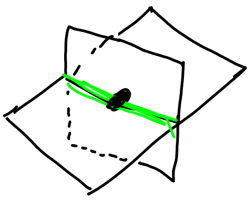
Lemma 3.1.2

Let L be an a-dim. lin. subspace of $\mathbb{P}^n$ and let M be a b-dim. lin. subspace of $\mathbb{P}^n$.
Then, $L \cap M$ is a c-dimensional lin. subspace of $\mathbb{P}^n$ with $c \geq a+b-n$. $\left(\underbrace{\mathrm{codim}(L\cap M, \mathbb{P}^n)}_{=\, n-c} \leq \underbrace{\mathrm{codim}(L, \mathbb{P}^n)}_{=\, n-a} + \underbrace{\mathrm{codim}(M, \mathbb{P}^n)}_{=\, n-b}\right)$

Ex If L, M are lines in $\mathbb{P}^2$, then $L \cap M$ is a point or $L = M$.



Pf of Lemma Let $L, M \subseteq \mathbb{P}^n_K$ corr. to $V, W \subseteq K^{n+1}$

$\dim(V) = a+1$, $\dim(W) = b+1$
$\mathrm{codim}(V, K^{n+1}) = n-a$, $\mathrm{codim}(W, K^{n+1}) = n-b$

$\Rightarrow$ $V \cap W$ is a vector space with
$\mathrm{codim}(V\cap W, K^{n+1}) \leq (n-a)+(n-b)$

$\Rightarrow$ $\dim(L \cap M) = n - \mathrm{codim}(V \cap W, K^{n+1}) \geq n-(n-a)-(n-b)$
$= a+b-n$.

$\square$

3.2. Algebraic sets

Def A polynomial $f \in K[X_0, \dots, X_n]$ is <u>homogeneous of degree $d \geq 0$</u> (or a <u>form of degree d</u>) if every monomial in f has degree (exactly) d.

<u>Ex</u> $2X + 3Y$ hom. of deg. 1

<u>Ex</u> $2X + 3Y + 1$ not hom.

<u>Ex</u> $X^3 + 2X^2Y + Y^3$ hom. of degree 3

<u>Ex</u> 0 is homogeneous of <u>every</u> degree $d \geq 0$.

<u>Rmk</u> The hom. degree d pol. form a K-vector space.

<u>Rmk</u> Any pol. $f \in K[X_0, \dots, X_n]$ can be written uniquely as $f = \sum_{d=0}^{m} f_d$ with f_d hom. of degree d (called the <u>degree d part</u> of f).

<u>Rmk</u> If f is hom. of degree d and g is hom. of degree e, then fg is hom. of degree $d+e$.

Rmk If $f \in K[Y_0,\dots,Y_m]$ is hom. of degree d and $g_1,\dots,g_m \in K[X_0,\dots,X_n]$ are hom. of degree e, then $f(g_1,\dots,g_m)$ is hom. of degree $d\cdot e$.

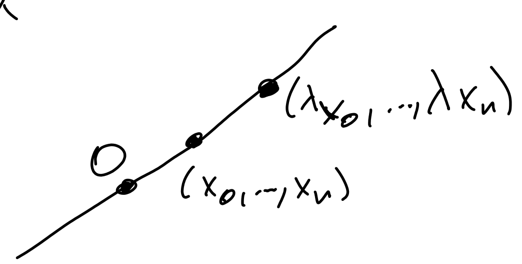
Rmk 3.2.1 If f is hom. of degree d, then

$$f(\lambda X_0,\dots,\lambda X_n) = \lambda^d f(X_0,\dots,X_n).$$

Def If $f \in K[X_0,\dots,X_n]$ is hom., we denote by

$$V_{\mathbb{P}^n_K}(f) = \{[x_0:\dots:x_n] \in \mathbb{P}^n_K \mid f(x_0,\dots,x_n) = 0\}$$

independent of the choice of hom. coord. $x_0,\dots,x_n$ by Rmk 3.2.1!



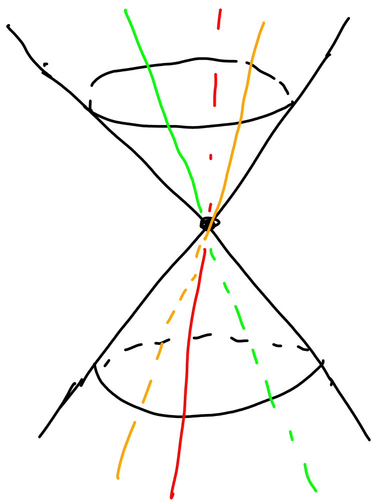
the corresponding set of zeros (the vanishing locus of f).

If $S \subseteq K[X_0,\dots,X_n]$ is a set of hom. pol., let

$$V_{\mathbb{P}^n_K}(S) = \{[x_0:\dots:x_n] \in \mathbb{P}^n_K \mid f(x_0,\dots,x_n) = 0 \ \forall f \in S\} = \bigcap_{f \in S} V_{\mathbb{P}^n_K}(f).$$

A subset $A = V_{\mathbb{P}^n_K}(S)$ of this form is called algebraic.

Ex $f = X_1^2 + X_2^2 - X_0^2$



$V(f) = V_{K^3}(f) =$ cone

$V_{\mathbb{P}^2}(f) =$ set of lines through O on the cone

Rmk $V_{\mathbb{P}^n_K}(S) \subseteq \mathbb{P}^n_K$ is the set of lines through O contained in $V(S) = V_{K^{n+1}}(S)$.

Ex Any linear subspace of $\mathbb{P}^n_K$ is algebraic.

Def Let $A \subseteq \mathbb{P}^n_K$ be any subset.

The set $C(A) = \{0\} \cup \{0 \neq (x_0, \dots, x_n) \in K^{n+1} \mid [x_0 : \dots : x_n] \in A\}$
$\subseteq K^{n+1}$

(the union of $\{0\}$ and the lines in K^{n+1} representing the points in $A \subseteq \mathbb{P}^n_K$)

is called the affine cone of A.

Lemma 3.2.2 If $A \subseteq \mathbb{P}^n_K$ is algebraic, then $\ell(A) \subseteq K^{n+1}$ is algebraic.

Pf If $A = V_{\mathbb{P}^n_K}(S) \neq \emptyset$, then $\ell(A) = V_{K^{n+1}}(S)$.

If $A = \emptyset$, then $\ell(A) = \{0\}$. $\square$

Rmk $\ell(A \cap B) = \ell(A) \cap \ell(B)$

As before (Lemma 2.2):

Rmk a) $\bigcap_\alpha V_{\mathbb{P}^n}(S_\alpha) = V_{\mathbb{P}^n}\left(\bigcup_\alpha S_\alpha\right)$

b) $V_{\mathbb{P}^n}(S) \cup V_{\mathbb{P}^n}(T) = V_{\mathbb{P}^n}(\{fg \mid f \in S,\ g \in T\})$

c) $V_{\mathbb{P}^n}(\emptyset) = V_{\mathbb{P}^n}(0) = \mathbb{P}^n$

d) $V_{\mathbb{P}^n}(1) = \emptyset$.

Hence, we again obtain a <u>Zariski topology</u> whose closed sets are the algebraic sets.

Ex Any affine patch $U \subseteq \mathbb{P}^n_K$ (complement of hyperplane) is open.

Lemma 3.2.3 Any affine chart $\varphi : K^n \to U \subseteq \mathbb{P}^n_K$ is continuous:

If $A \subseteq \mathbb{P}^n_K$ is algebraic, then $\varphi^{-1}(A) \subseteq K^n$ is algebraic.

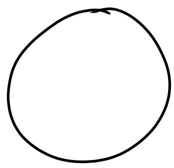
Pf $\varphi^{-1}(A) \subseteq K^n$ "is" the intersection of the affine cone $\mathcal{C}(A)$ with the affine linear subspace T corresponding to the chart. $\square$

Concretely If $\varphi = \varphi_i$ is the i-th standard affine chart, $A = V_{\mathbb{P}^n}(S)$, then

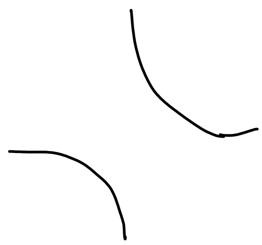
$$\varphi^{-1}(A) = \{(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in K^n \mid f(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) = 0 \ \forall f \in S\}.$$

Ex $A = V_{\mathbb{P}^2}(x_1^2 + x_2^2 - x_0^2)$.

$$\varphi_0^{-1}(A) = V_{K^2}(x_1^2 + x_2^2 - 1)$$



$$\varphi_1^{-1}(A) = V_{K^2}(1 + x_2^2 - x_0^2)$$



The preimage $\varphi^{-1}(A)$ is a conic section for any affine chart φ.

We constructed a map

$$\{\text{alg. subset } A \text{ of } \mathbb{P}^n\} \longrightarrow \{\text{alg. subset } B \text{ of } K^n\}$$

$$A \longmapsto \varphi^{-1}(A)$$

Q How to produce $A \subseteq \mathbb{P}^n_K$ from $B \subseteq K^n$?

A Take $A = \overline{\varphi(B)}$. What are equations defining A?

Def Let $f \in K[X_1, \dots, X_n]$ be a polynomial of degree d and let f_e be its degree e (hom.). The

homogenization of $f = \sum_e f_e$ (at X_0) is the hom. degree d pol.

$$\tilde{f} = \sum_e f_e \cdot X_0^{d-e} = X_0^d f\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right)$$

Ex $f = x_1^2 + x_2^2 - 1 \rightsquigarrow \tilde{f} = x_1^2 + x_2^2 - x_0^2$

Note $\tilde{f}(1, x_1, \dots, x_n) = f(x_1, \dots, x_n)$

Lemma 3.2.4 Let $B = V_{K^n}(I)$ for an ideal $I \subseteq K[X_1, \dots, X_n]$. Let $\varphi = \varphi_0$ be the 0-th standard chart of $\mathbb{P}^n_K$. Then,

$\overline{\varphi(B)} = V_{\mathbb{P}^n_K}(S)$ where $S \subseteq K[X_0, \dots, X_n]$ is the set of homogenizations $\tilde{f}$ of the elements $f \in I$ at X_0.

Pf HW.

Cor 3.2.5

$$\varphi^{-1}\left(\overline{\varphi(B)}\right) = B \text{ for any affine chart.}$$

(We only add points at ∞ to B to obtain $\overline{\varphi(B)}$.)

Ex $B = \{(x_1, x_2) \mid x_1 x_2 = 1\}$

$$\rightsquigarrow A = \overline{\varphi_0(B)} = \{[x_0 : x_1 : x_2] \mid x_1 x_2 = x_0^2\}$$

the two pts. at ∞

What are the points at ∞?

pt. at ∞

$$A \setminus \varphi_0(B) = \{[x_0 : x_1 : x_2] \mid x_1 x_2 = x_0^2,\ x_0 = 0\}$$

$$= \{[0 : x_1 : x_2] \mid x_1 x_2 = 0\}$$

$$= \{[0:0:1], [0:1:0]\}$$

Warning Let $I = (f_1, \ldots, f_m)$.

Then, S is the set of homogenizations of elements of I. Unfortunately, the homogenizations $\tilde{f}_1, \ldots, \tilde{f}_m$ don't always suffice!

Ex $I = (x_1^2 + x_2, x_1) = (x_2, x_1)$

$$x_1^2 + x_0 x_2 = 0, \ x_1 = 0 \qquad\qquad x_2 = 0, \ x_1 = 0$$

$$\Updownarrow$$

$$x_0 x_2 = 0, \ x_1 = 0$$

two points

$[0:0:1], [1:0:0]$

one point

$[1:0:0]$