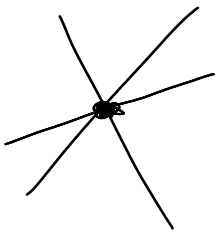
In this section, K can be any field (not necc. alg. closed).

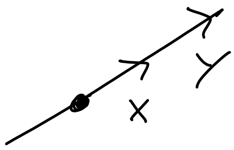
Def The n-dimensional <u>projective space</u> $\mathbb{P}^n_K$ over K is the set of lines in K^{n+1} through the origin. We call the elements of $\mathbb{P}^n_K$ the <u>points</u> in $\mathbb{P}^n_K$.



We denote the line spanned by $(0,\dots,0) \in (x_0,\dots,x_n) \in K^{n+1}$ by $[x_0 : \dots : x_n] \in \mathbb{P}^n_K$.

<u>Note</u> $[x_0 : \dots : x_n] = [y_0 : \dots : y_n]$ if and only if $(x_0,\dots,x_n), (y_0,\dots,y_n) \in K^{n+1}$ are colinear, i.e.

$$(x_0,\dots,x_n) = \lambda(y_0,\dots,y_n) \text{ for some } \lambda \in K^\times.$$



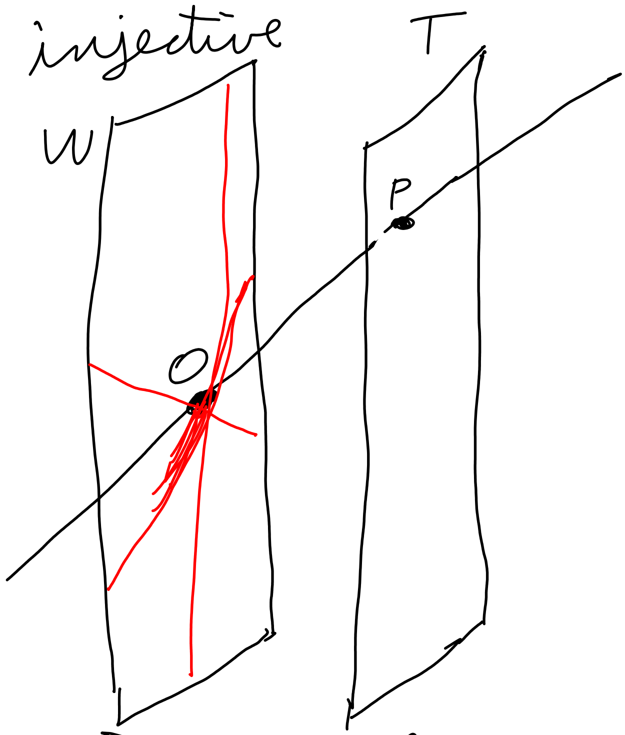
$x_0,\dots,x_n$ are called <u>projective coordinates</u> of the point $[x_0 : \dots : x_n] \in \mathbb{P}^n_K$.

Rmk We could therefore equivalently have defined $\mathbb{P}^n_K$ to be the set of $(n+1)$-tuples $(0,\dots,0) \neq (x_0, \dots, x_n) \in K^{n+1}$ modulo the following equivalence relation:

$$(x_0, \dots, x_n) \sim (y_0, \dots, y_n) \text{ if } (x_0, \dots, x_n) = \lambda (y_0, \dots, y_n) \text{ for some } \lambda \in K^\times.$$

In short: $\mathbb{P}^n_K = \left(K^{n+1} \setminus \{0\}\right) / K^\times$.

Rmk For any n-dimensional affine linear subspace $T \subset K^{n+1}$ not containing the origin, we have an injective map



$$T \hookrightarrow \mathbb{P}^n_K$$

$P \longmapsto$ line spanned by P

$(x_0, \dots, x_n) \mapsto [x_0 : \dots : x_n]$

Its image $U \subset \mathbb{P}^n_K$ (consisting of the lines in K^{n+1} intersecting T) is an <u>affine patch</u> of $\mathbb{P}^n_K$.

For a choice of linear bijection $T \cong K^n$, we obtain a bijection φ between $\mathbb{A}^n_K = K^n$ and U called a chart (map) of $\mathbb{P}^n_K$.

Ex For $i = 0, \dots, n$, we can take

$$T_i = \{ (x_0, \dots, x_n) \in K^{n+1} \mid x_i = 1 \}$$

and the i-th standard chart (map)

$$\varphi_i : K^n \hookrightarrow \mathbb{P}^n_K$$

$$(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \mapsto [x_0 : \dots : x_{i-1} : 1 : x_{i+1} : \dots : x_n]$$

$$\left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right) \mapsto [x_0 : \dots : x_n]$$

with image $U_i = \{ [x_0 : \dots : x_n] \in \mathbb{P}^n_K \mid x_i \neq 0 \}$.

$$\mathbb{P}^n_K \setminus U_i = \{ [x_0 : \dots : x_n] \mid x_i = 0 \}$$

$$\cong \{ [x_0 : \dots : x_{i-1} : x_{i+1} : \dots : x_n] \} = \mathbb{P}^{n-1}_K.$$

Rmk More generally, the complement of U in $\mathbb{P}^n_K$ consists of the lines in K^{n+1} through 0 that are parallel to T, i.e. that lie in the n-dimensional linear subspace W of K^{n+1} parallel to T.

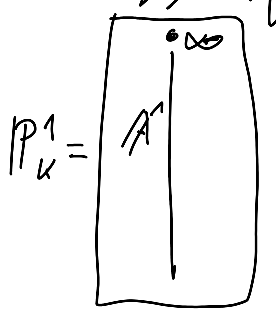
$\Rightarrow$ Identifying W with K^n, we obtain a bijection $\mathbb{P}^n_K \setminus \underbrace{U}_{\mathbb{A}^n_K} \cong (\text{lines through } 0 \text{ in } K^n) \cong \mathbb{P}^{n-1}_K$

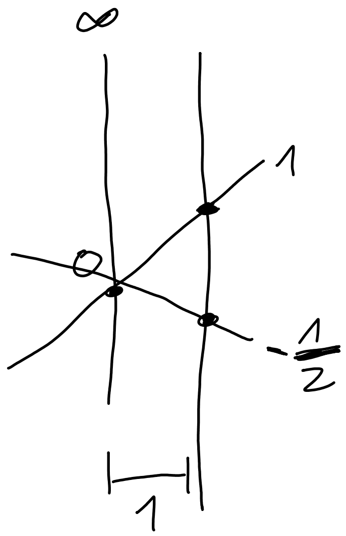
$$\rightsquigarrow \mathbb{P}^n_K = \mathbb{A}^n_K \sqcup \underbrace{\mathbb{P}^{n-1}_K}_{\text{"set of points at } \infty\text{"}}$$

<u>Ex</u> $\mathbb{P}^0_K = \{\text{lines through } 0 \text{ in } K^1\} = \{*\}$

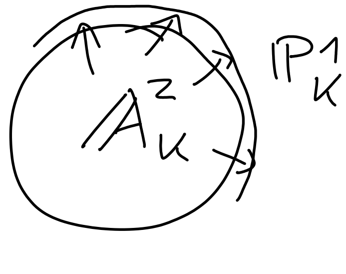
single pt.

$$\mathbb{P}^1_K = \mathbb{A}^1_K \sqcup \{\infty\}$$





$$\mathbb{P}^2_K = \mathbb{A}^2_K \sqcup \underbrace{\mathbb{P}^1_K}_{\text{pts at } \infty}$$

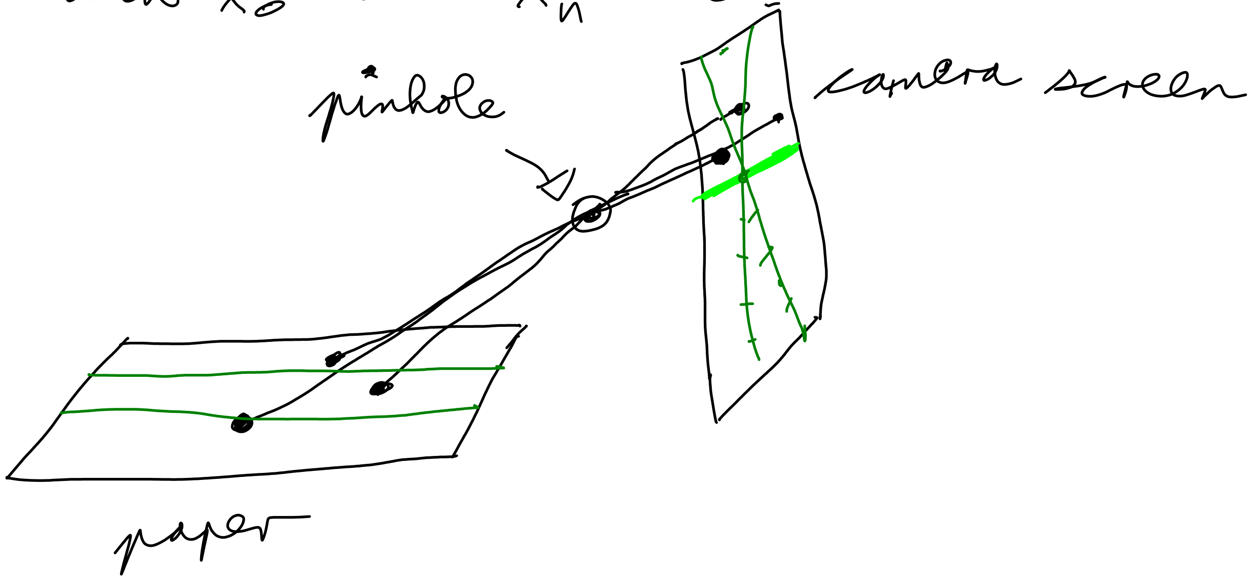


Rmk The standard affine patches $U_0, \dots, U_n$ cover $\mathbb{P}^n_k$: $\mathbb{P}^n_k = \bigcup_{i=0}^n U_i$.

Def $U_i = \{[x_0 : \dots : x_n] \mid x_i \neq 0\}$.

$\Rightarrow \bigcup_{i=0}^n U_i = \{[x_0 : \dots : x_n] \mid x_i \neq 0 \text{ for some } i\}$

But there is (by def.) no point $[x_0 : \dots : x_n] \in \mathbb{P}^n_k$ with $x_0 = \dots = x_n = 0$. $\square$



Def A d-dimensional <u>linear subspace</u> L of $\mathbb{P}^n_k$ is the set of lines through 0 contained in a fixed $(d+1)$-dimensional linear subspace V of k^{n+1}.

Rmk Identifying V with k^{d+1}, we obtain a bijection $L \cong \mathbb{P}^d_k$.

Ex 0-dim. lin. subsp. of $\mathbb{P}^n$

$=$ single point in $\mathbb{P}^n_K$

Ex 1-dim. lin. subsp. are called **lines** in $\mathbb{P}^n_K$.

Ex 2 **planes**

Ex $(n-1)$ **hyperplanes**

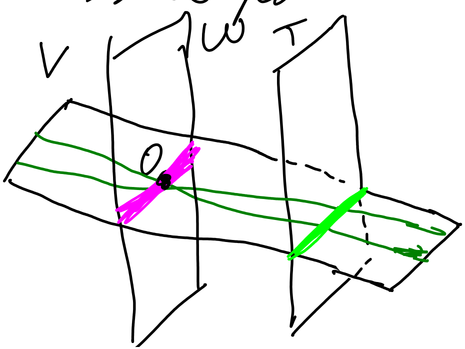
($\mathbb{P}^n_K \setminus U$ as above is a hyperplane in $\mathbb{P}^n_K$.)

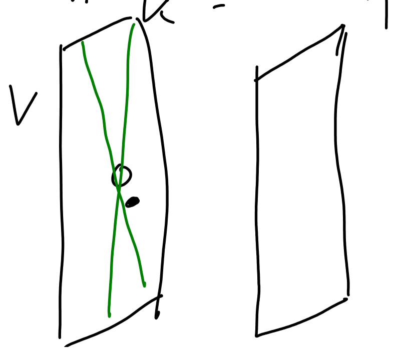
Ex (-1)-dim. lin. subsp. $= \emptyset$.

Lemma 3.1.1 Let $\varphi: K^n \xrightarrow{\sim} U \subset \mathbb{P}^n_K$ be an affine chart and let $L \subseteq \mathbb{P}^n_K$ be a d-dimensional linear subspace. Then, **either**

a) $\varphi^{-1}(L) \subseteq K^n$ is an affine d-dimensional linear subspace and $L \cap (\mathbb{P}^n_K \setminus U)$ is a $(d-1)$-dimensional linear subspace of $\mathbb{P}^n_K \setminus U \cong \mathbb{P}^{n-1}_K$.

or b) $\varphi^{-1}(L) = \emptyset$ and L is a d-dimensional linear subspace of $\mathbb{P}^n_K \setminus U \cong \mathbb{P}^{n-1}_K$.

a) 

b) 

Pf Let $W \subset K^{n+1}$ be the n-dim. lin. subsp. of K^{n+1} parallel to the affine lin. subsp. $T \subset K^{n+1}$ defining the affine chart.

If $V \subseteq W$, then $V \cap T = \emptyset$, so $\varphi^{-1}(L) = \emptyset$.

$\Rightarrow$ b).

If $V \not\subseteq W$, then $V + W = K^{n+1}$, so

$$\dim(V \cap W) = \dim(V) + \dim(W) - \dim(V+W)$$
$$= (d+1) + n - (n+1) = d$$

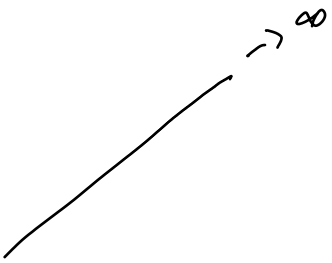
and $\underset{\substack{\Uparrow \\ T = W + s \text{ for some } s \in K^{n+1} \\ = W + v \quad \overset{\shortparallel}{v + w} \\ \text{for some } v \in V,\, w \in W}}{V \cap T \neq \emptyset}$ is a translate of $V \cap W$

$\Rightarrow$ a) $\square$

Rmk Conversely, every d-dimensional affine linear subspace ℓ of K^n corr. to exactly one d-dimensional linear subspace L of $\mathbb{P}^n_K$, namely the set of lines through 0 contained in the subspace V of K^{n+1} spanned by the lines $\varphi(P)$ for $P \in \ell$.

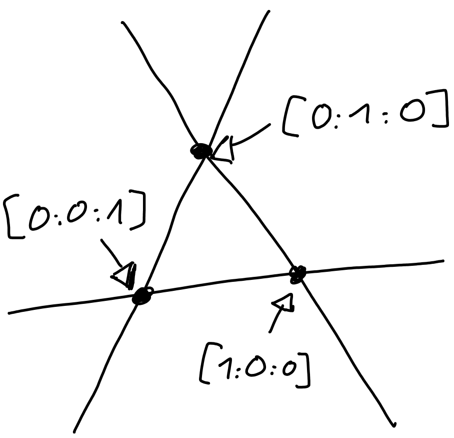
<u>Eg</u> The lines in $\mathbb{P}^2_k = \mathbb{A}^2_k \sqcup \mathbb{P}^1_k$ "are"

- The lines in $\mathbb{A}^2$ (with one point at ∞ each)



- The line $\mathbb{P}^1_k$ at ∞.

$$\mathbb{P}^2 \setminus U_0 = \{[x_0 : x_1 : x_2] \mid x_0 = 0\}$$



$$\mathbb{P}^2 \setminus U_1 = \{[x_0 : x_1 : x_2] \mid x_1 = 0\}$$

$$\mathbb{P}^2 \setminus U_2 = \{[x_0 : x_1 : x_2] \mid x_2 = 0\}$$