Pf of Thm 2.74 (Going down)

φ^* gives an inclusion $\Gamma(W) \hookrightarrow \Gamma(V)$ and an inclusion $K(W) \hookrightarrow K(V)$.

W normal: $\Gamma(W)$ is integrally closed in $K(W)$.

φ finite: $\Gamma(V)$ is a module-finite ring ext. of $\Gamma(W)$

$K(V)$ is a finite field ext. of $K(W)$.

Let L be the normal closure of this field ext.
It's a finite field ext. of $K(W)$.

$$\begin{tikzcd}
L \arrow[d, dash] & \supseteq & S = \Gamma(V') \arrow[d, dash] & V' \arrow[d, "\psi"] & A_1' \cup \dots \cup A_s' \arrow[d] \\
K(V) \arrow[d, dash] & \supseteq & \Gamma(V) \arrow[d, dash] & V \arrow[d, "\varphi"] & A_1 \cup \dots \cup A_r \arrow[d] \\
K(W) & \supseteq & \Gamma(W) & W & B
\end{tikzcd}$$

Let S be the integral closure of $\Gamma(W)$ in L.
Since $\Gamma(V)$ is an integral ext. of $\Gamma(W)$, we have $\Gamma(V) \subseteq S$.
We have $S \cap K(W) = \Gamma(W)$ because $\Gamma(W)$ is integrally closed in $K(W)$.
S is a module-finite ring ext. of $\Gamma(W)$ because it is integral and L is a finite field ext of $K(W)$.

$\Gamma(W) = K[x_1, \ldots]/\ldots$ is a finitely generated ring ext. of K.

$\Rightarrow S$ is a finitely generated ring ext. of K

$$S = K[y_1, \ldots]/\ldots$$

$\Rightarrow S$ corresponds to an irreducible algebraic set V' with $\Gamma(V') = S$.

$\Gamma(V') = S$ is an integral mod.-fin. ext of $\Gamma(W)$ and therefore also $\Gamma(V)$.

The inclusion $\Gamma(V) \subseteq \Gamma(V')$ corresponds to a dominant finite morphism $\psi : V' \to V$.

Decompose $(\varphi \circ \psi)^{-1}(B)$ into irreducible components: $(\varphi \circ \psi)^{-1}(B) = A'_1 \cup \ldots \cup A'_s$.

<u>Claim</u> Any A_i contains $\psi(A'_j)$ for some j.

<u>Pf</u> W.l.o.g. $i = 1$. Let $P \in A_1 \setminus (A_2 \cup \ldots \cup A_r)$ (exists because $A_1 \not\subseteq A_2 \cup \ldots \cup A_r$).

Since ψ is dom. + fin., it's surjective.

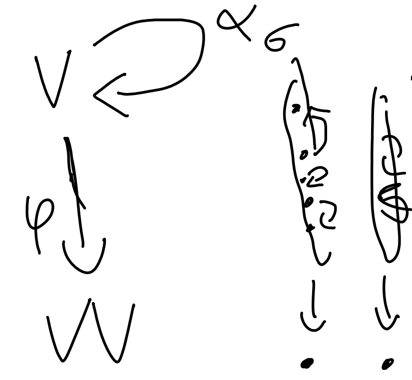
Let $P' \in \psi^{-1}(P)$. Let $P' \in A'_j$.

$\psi(A'_j)$ is an irred. subset of $\varphi^{-1}(B) = A_1 \cup \ldots \cup A_r$.

$\Rightarrow P \in \psi(A'_j) \subseteq A_i$ for some $i \underset{P \notin A_2, \ldots, A_r}{\Rightarrow} i = 1 \Rightarrow \psi(A'_j) \subseteq A_1$ $\square$

$\Rightarrow$ It suffices to show that $\varphi(\psi^{-1}(A_j)) = B$ for all j.

$\Rightarrow$ We can assume w.l.o.g. that $K(V)$ is a normal field ext. of $K(W)$ and $V' = V$, $L = K(V)$, $S = \Gamma(V)$.



Let $G = Gal(K(V)|K(W))$ be the group of automorphisms of $K(V)$ fixing every element of $K(W)$.

Note: If $a \in K(V)$ satisfies a monic polynomial equation with coefficients in $\Gamma(W)$, then $\sigma(a) \in K(V)$ satisfies the same equation for any $\sigma \in G$

$\Rightarrow \sigma(\Gamma(V)) \subseteq \Gamma(V) \quad \forall \sigma \in G$.

$\Rightarrow$ The automorphism σ of $K(V)$ restricts to a ring automorphism of $\Gamma(V)$ fixing every element of $\Gamma(W)$.

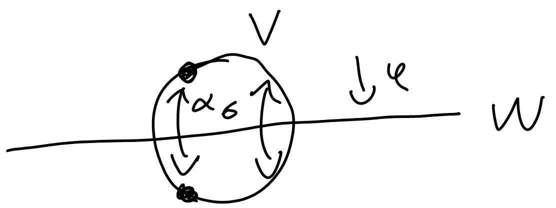
The hom. $\sigma : \Gamma(V) \longrightarrow \Gamma(V)$ corresponds to a morphism $\alpha_\sigma : V \longrightarrow V$ (with $\alpha_\sigma^* = \sigma$).

Since σ fixes $\Gamma(W)$, we have comm. diagrams

$$\begin{tikzcd} & \Gamma(V) \arrow[d, "\sigma"] \\ \Gamma(W) \arrow[ur, hook] \arrow[r, hook] & \Gamma(V) \end{tikzcd} \qquad \begin{tikzcd} & V \arrow[dl, "\varphi"'] \\ W & V \arrow[u, "\alpha_\sigma"'] \arrow[l, "\varphi"] \end{tikzcd}$$

($\leadsto$ "deck transformation")

Ex Say $K=\mathbb{C}$, $V=\{(x,y)\in\mathbb{C}^2 \mid x^2+y^2=1\}$, $W=K$,
$\varphi(x,y)=x$.



$\Gamma(W)=K[x]$, $\quad \Gamma(V)=K[x,y]/(x^2+y^2-1)=K[x][\underbrace{\sqrt{1-x^2}}_{y}]$

$K(W)=K(x)$, $\quad K(V)= \quad \ldots \quad =K(x)(\sqrt{1-x^2})$

$K(V)$ is a Galois ext. of $K(W)$ with Galois group $\{id,\sigma\}$ where $\sigma(\underbrace{\sqrt{1-x^2}}_{y})=\underbrace{-\sqrt{1-x^2}}_{-y}$

corresponds to the reflection of V across the x-axis.

Then, $\alpha_\sigma(\varphi^{-1}(B))=\varphi^{-1}(B)$, so α_σ permutes the irreducible components $A_1,\ldots,A_r$ of $\varphi^{-1}(B)$.

Claim G acts transitively on the set of irred. components.

Pf Assume w.l.o.g. $A_1,\ldots,A_l$ lie in different G orbits than $A_{l+1},\ldots,A_r$.

Pick points $P_1\in A_1,\ldots,P_l\in A_l$ with $P_1,\ldots,P_l\notin A_{l+1}$. By the Chinese remainder

theorem, since $\{P_1\}, \dots, \{P_c\}$, A_{c+1} are pairwise disjoint, there is a function
$f \in \Gamma(V)$ with $f(P_1) = \dots = f(P_c) = 1$ and
$f|_{A_{c+1}} = 0$.

$f(P_i) = 1 \Rightarrow f|_{A_i} \neq 0$ for $i = 1, \dots, c$.

We have

$$g := N_{K(V)|K(W)}(f) = \left(\prod_{\sigma \in G} \sigma(f)\right)^t,$$

where $t \geq 1$ is the degree of inseparability of $K(V)|K(W)$.

$$g \in \Gamma(V) \cap K(W) \underset{\uparrow}{=} \Gamma(W)$$

$\Gamma(W)$ int. closed in $K(W)$

(I) $g|_{A_j} = \left(\prod \sigma(f)\right)\Big|_{A_j} \neq 0$ because $\Gamma(A_j)$ is an integral domain and $f|_{A_1}, \dots, f|_{A_c} \neq 0$ and α_σ permutes just $A_1, \dots, A_c$.

On the other hand,

$$g|_{A_{c+1}} = \Big(\underbrace{\prod \sigma(f)}_{f \cdot \dots}\Big)\Big|_{A_{c+1}} = 0.$$

Since g is (the composition with φ of) a function on W, we have

$$g|_{\varphi(A_{l+1})} = 0.$$

W.l.o.g. $\varphi(A_{l+1}) = B$ by Lemma 2.73.

$$\Rightarrow g|_B = 0$$

On the other hand, (I) says that

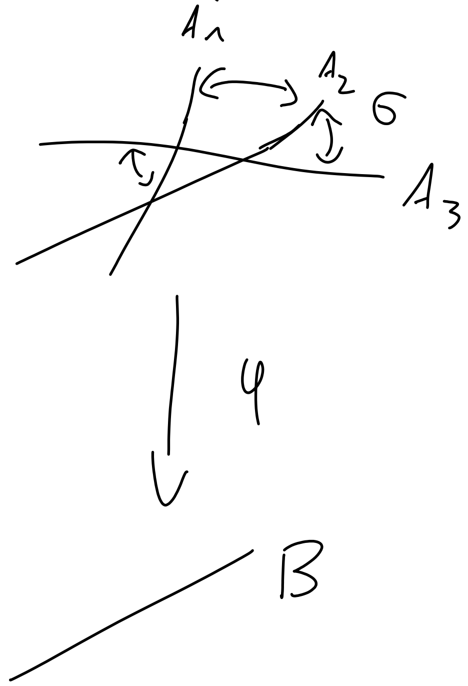
$$g|_{\underbrace{\varphi(A_i)}_{\subseteq B}} \neq 0 \quad \text{for } i = 1, \ldots, l. \quad \lightning$$

$\square$

G acts transitively on the irred. comp. $A_1, \ldots, A_r$ of $\varphi^{-1}(B)$ and $\varphi(A_i) = B$ for some i. If $\alpha_G(A_i) = A_j$, then

$$\varphi(A_j) = \varphi(\alpha_G(A_i)) = \varphi(A_i) = B.$$

$\square$

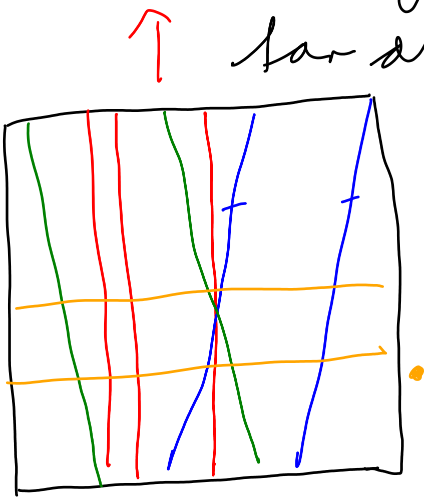


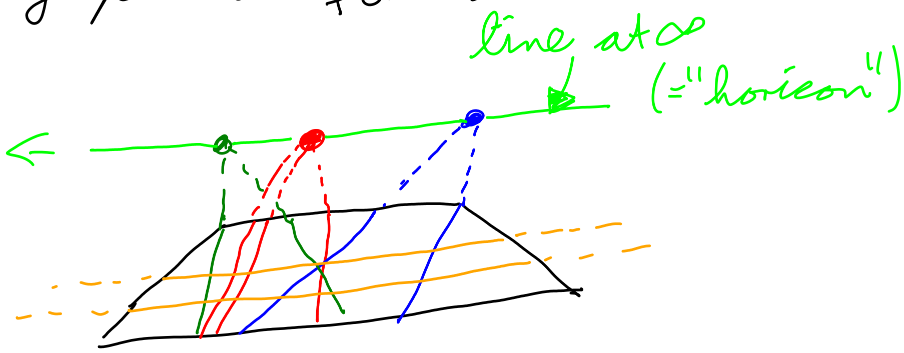
3. Projective varieties

3.1. Projective space

Two lines $L_1 \neq L_2$ in $\mathbb{R}^2$ intersect in exactly one point except if they are parallel.

Idea: Pretend they intersect in a point at infinity by adding one infinitely far away point for each direction.



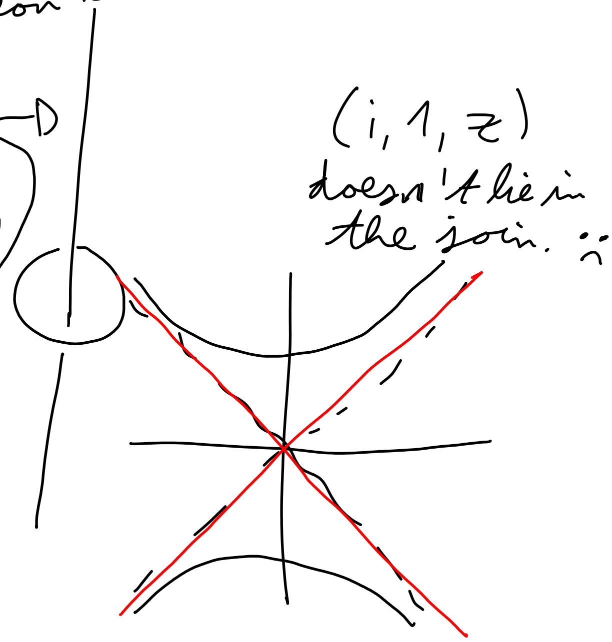


Any line goes through one point at ∞ (corr. to its direction).

Matty's question on the double-bonus problem:

works for $\mathbb{R}3$ but not for $\mathbb{C}3$!

$(i, 1, z)$ doesn't lie in the join :(



$$x^2+y^2=1$$

$$x' = ix, \quad y' = y$$

$$-x'^2+y'^2=1$$

$$= (y'-x')(x'+y')$$