Rmk For any $m \geq 2$, there are points $P_1, \dots, P_m \in K^2$ s.t. there is no irreducible $0 \neq f \in K[X,Y]$ of degree $\leq m-2$ with $P_1, \dots, P_m \in V(f)$.

Pf Take $P_1, \dots, P_{m-1}$ on x-axis, P_m not on the x-axis.

The restriction $f(X, 0)$ of f to the x-axis is a pol. of degree $\leq m-2$ with $\geq m-1$ roots. $\Rightarrow$ It's the zero polynomial.

$\Rightarrow f = Y \cdot g$ for some pol. g.

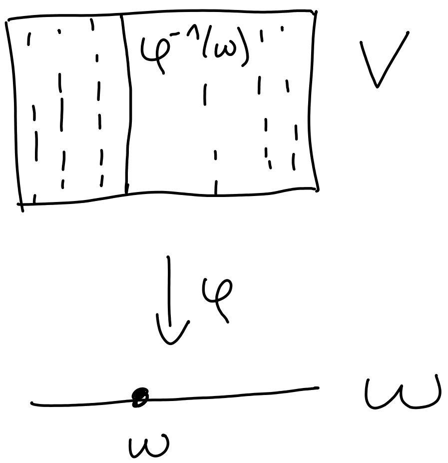
But $g \neq$ const. because $f(P_m) = 0$.

$\Rightarrow f$ is reducible.

$\square$

2.21. Dimensions of fibers

Def A *fiber* of $\varphi: V \to W$ is the preimage $\varphi^{-1}(w)$ of a point $w \in W$.



Thm 2.87 Let V, W be irreducible, $\varphi: V \to W$ a morphism, and A an irreducible component of $\varphi^{-1}(w)$ for some point $w \in W$. Then,

$$\operatorname{codim}(A, V) \leq \dim(W).$$

$$(\dim(A) \geq \dim(V) - \dim(W).)$$

In particular,

$$\dim(\varphi^{-1}(w)) \geq \dim(V) - \dim(W)$$ for every $w \in \varphi(V)$.

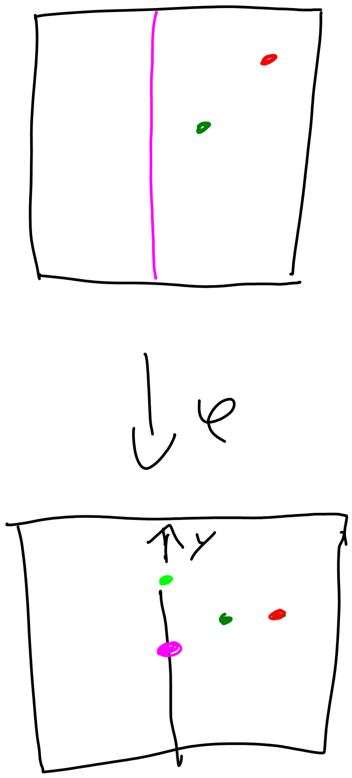
<u>Ese</u> $\varphi : K^2 \longrightarrow K^2$

$$(x,y) \longmapsto (x, xy)$$

$\varphi^{-1}(a,b) = \left\{\left(a, \frac{b}{a}\right)\right\}$ if $a \neq 0$

$\varphi^{-1}(0,b) = \emptyset$ if $b \neq 0$

$\varphi^{-1}(0,0) = \{(0,y) \mid y \in K\}$



We'll actually prove something more general:

<u>Thm 2.88</u> Let V, W, φ as above, $B \subseteq W$ irreducible, and A an irreducible component of $\varphi^{-1}(B)$ with $\overline{\varphi(A)} = B$. Then, $\operatorname{codim}(A, V) \leq \operatorname{codim}(B, W)$.

<u>Rmk</u> If $B = \{w\}$, then automatically $\varphi(A) = \{w\}$.

<u>Rmk</u> The condition $\overline{\varphi(A)} = B$ can't be omitted. (Look at the example just before the def. of normal alg. sets.)

Pf Let $n = \operatorname{codim}(B, W)$.

By Cor. 2.80, there are functions $g_1, \dots, g_n \in \Gamma(W)$ s.t. B is an irred. comp. of $V(g_1, \dots, g_n) \subseteq W$.

$$\Rightarrow \underset{\text{irred.}}{A} \subseteq \varphi^{-1}(B) \subseteq \varphi^{-1}(V(g_1, \dots, g_n)) = V(\underbrace{\varphi^*(g_1)}_{= f_1}, \dots, \underbrace{\varphi^*(g_n)}_{= f_n})$$

$\Rightarrow$ A is contained in some irred. comp. A' of $V(f_1, \dots, f_n)$

$$\Rightarrow \underset{\substack{\text{irred.} \\ \text{comp. of} \\ V(g_1, \dots, g_n)}}{B} = \overline{\varphi(A)} \subseteq \underset{\text{irred.}}{\overline{\varphi(A')}} \subseteq V(g_1, \dots, g_n)$$

$$\Rightarrow B = \overline{\varphi(A')}$$

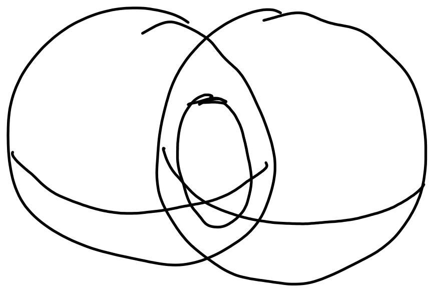
$$\Rightarrow \underset{\substack{\text{irred.} \\ \text{comp. of} \\ \varphi^{-1}(B)}}{A} \subseteq \underset{\text{irred.}}{A'} \subseteq \varphi^{-1}(B)$$

$\Rightarrow A = A'$, which is an irred. comp. of $V(f_1, \dots, f_n)$

Thm 2.83 $\Rightarrow \operatorname{codim}(A, V) \leq n$. $\square$

<u>Cor 2.89</u> Let $V_1, V_2 \subseteq W$ all be irred. and let A be an irred. comp. of $V_1 \cap V_2$.

Then, $\operatorname{codim}(A, W) \leq \operatorname{codim}(V_1, W) + \operatorname{codim}(V_2, W)$.



<u>Pf</u> Consider the inclusion morphism

$$\varphi : V_1 \longrightarrow W.$$

We have $\varphi^{-1}(V_2) = V_1 \cap V_2$.

$$\Rightarrow \underbrace{\operatorname{codim}(A, V_1)}_{\operatorname{codim}(A, W) - \operatorname{codim}(V_1, W)} \leq \operatorname{codim}(V_2, W) \qquad \square$$

If φ is dominant, we have equality for a "generic" fiber.

<u>Prop 2.90</u> Let V, W, φ as above and assume φ is dominant. Then, there is an open $\emptyset \neq U \subseteq W$ contained in $\varphi(V)$ and such that every irred. comp. A of every fiber $\varphi^{-1}(w)$ with $w \in U$ satisfies $\operatorname{codim}(A, V) = \dim(W)$.

We won't prove this.

2.22. Applications of dimension, part 2

We obtain a "converse" of Thm 2.85:

<u>Thm 2.91</u> For any $n, d \geq 1$ and $m \geq \binom{d+n}{n}$,

then there are m points $P_1, \dots, P_m \in K^n$ such that there is no nonzero polynomial $0 \neq f \in K[x_1, \dots, x_n]$ of degree $\leq d$ with $P_1, \dots, P_m \in V(f)$.

<u>Pf</u> Let F_d^1 be the set of pol. of degree $\leq d$ where at least one coeff. is 1.

$\dim(F_d^1) = \binom{d+n}{n} - 1$

Consider the following algebraic subset A of $K^n \times \dots \times K^n \times F_d^1$:

$$A = \{(P_1, \dots, P_m, f) \mid f(P_1) = \dots = f(P_m) = 0\}$$

~~For~~ an irred. comp. A' of A, consider the projection $\pi : A' \longrightarrow F_d^1$. Its image is contained in the irred. set $\overline{\pi(A')}$. We'll apply Thm 2.87 to $\pi : A' \to \overline{\pi(A')}$. Pick any $f \in \pi(A')$. Its preimage is

$$\pi^{-1}(f) = \underbrace{V(f) \times \dots \times V(f)}_{m} \times \{f\}.$$

$$\Rightarrow \dim(\pi^{-1}(f)) = m \cdot \dim(V(f))$$
$$= m \cdot (n-1)$$
$$f \neq 0$$

On the other hand, by Thm 2,87,

$$\dim(\pi^{-1}(f)) \geq \dim(A') - \dim(\overline{\pi(A')})$$
$$\geq \dim(A') - \dim(F_d')$$
$$= \dim(A') - \left(\binom{d+n}{n} - 1\right).$$

$$\Rightarrow \dim(A') \leq m(n-1) + \binom{d+n}{n} - 1$$
$$\leq mn - 1.$$
$$m \geq \binom{d+n}{n}$$

Since this holds for every irred. comp. A' of A, $\dim(A) \leq mn-1$.

$\Rightarrow$ The projection $A \longrightarrow \underbrace{K^n \times \dots \times K^n}_{m}$ is not surjective.

$\Rightarrow$ There are points $P_1, \dots, P_m \in K^n$ such that there is no $f \in F_d'$ with

$$f(P_1) = \dots = f(P_m) = 0.$$

$\square$

2.23. Some promised proofs

Pf of Lemma 2.81

Let $f \in R[X]$ be a monic pol. with $f(a) = 0$ and let $g \in K[X]$ be the min. polynomial of a.

$\Rightarrow g \mid f$

$\Rightarrow$ Every root of g in $\overline{K}$ is a root of f and therefore integral over R.

Write $g(X) = \prod_i (X - \underbrace{a_i}_{\in \overline{K}}) = X^n + c_{n-1}X^{n-1} + \dots + c_0$.

every coeff. $c_i \in K$ is $\pm$ a sum of products of roots, and therefore integral over R. Since R is integrally closed in K, this means that $c_i \in R$.

$$\Rightarrow b = \mathrm{Nm}_{L|K}(a) = \mathrm{Nm}_{K(a)|K}(\mathrm{Nm}_{L|K(a)}(a))$$
$$= \mathrm{Nm}_{K(a)|K}\left(a^{[L:K(a)]}\right) = \mathrm{Nm}_{K(a)|K}(a)^{[L:K(a)]}$$
$$= (\pm c_0)^{[L:K(a)]} \in R.$$

Furthermore, $0 = g(a) = a^n + c_{n-1}a^{n-1} + \dots + c_0$, so

$$a(a^{n-1} + \underbrace{c_{n-1}}_{\in R}a^{n-2} + \dots + \underbrace{c_1}_{\in R}) = -c_0 \in R \Rightarrow a \mid c_0 \mid b \text{ in } S.$$

$S \ni$

$\square$