2.19. Subsets defined by few equations

Lemma 2.81 Let S be a module-finite ring extension of R and assume that S, R are integral domains with fields of fractions L, K (K not necessarily alg. closed).
Let $0 \neq a \in S$ and $b := \mathrm{Nm}_{L|K}(a) \in K$, where the <u>norm map</u> $\mathrm{Nm}_{L|K} : L \longrightarrow K$ sends $a \in L$ to the determinant of the K-linear map $L \longrightarrow L$ sending x to ax. Assume that R is integrally closed in K.

$$\begin{array}{ccc} S & \subseteq & L \\ | & & | \\ R & \subseteq & K \end{array}$$

Then, $b \in R$ and $a \mid b$ in S.

<u>Pf</u> later ...

(If $L|K$ is Galois, then $\mathrm{Nm}_{L|K}(a) = \prod_{\sigma \in \mathrm{Gal}(L|K)} \sigma(a)$.)

Thm 2.82 (Krull's principal ideal theorem)

Let W be an irreducible alg. set and V be an irred. subset of $V(f) \subseteq W$ for $0 \neq f \in \Gamma(W)$.

Then, $\operatorname{codim}(V, W) = 1$ (so $\dim(V) = \dim(W) - 1$).

Pf Let $\varphi : W \longrightarrow K^n$ be a dominant finite morphism.

Goal: Find $0 \neq g \in K[X_1, \dots, X_n]$ s.t.

$$\varphi(V(f)) = V(g).$$

Then, $\dim(V) = \dim(\varphi(V(f))) = \dim(V(g))$
$= n - 1$.

Consider the field ext. $K(W) \mid \varphi^*(K(X_1, \dots, X_n))$.

We get a norm map

$$Nm : K(W) \longrightarrow \varphi^*(K(X_1, \dots, X_n)) \cong K(X_1, \dots, X_n).$$

Let $0 \neq g = Nm(f) \in K(X_1, \dots, X_n)$.

Then, g is integral over $K[X_1, \dots, X_n]$, so in fact $g \in K[X_1, \dots, X_n]$ (because $K[X_1, \dots, X_n]$ is integrally closed in $K(X_1, \dots, X_n)$ by Thm 2.15.)

Furthermore $\varphi^*(g) \mid f$ in $\Gamma(W)$, so

$V(f) \subseteq V(\varphi^*(g))$ and therefore $V(f) \subseteq V(\varphi^*(g))$,

so $\varphi(V(f)) \subseteq \varphi(\underbrace{V(\varphi^*(g))}_{\varphi^{-1}(V(g))}) \subseteq V(g)$.

Since $\varphi(V(f)) \subseteq V(g)$ is an algebraic set, if $\varphi(V(f)) \subsetneq V(g)$, there would exist some $h \in K[X_1, \dots, X_n]$ with $h|_{\varphi(V(f))} = 0$ but $h|_{V(g)} \neq 0$.

$$\varphi^*(h)|_{V(f)} = 0$$

$$\Downarrow \text{ Nullstellensatz}$$

$$\varphi^*(h)^m \in (f) \subseteq \Gamma(W) \quad \text{for some } m \geq 1$$

$$\Updownarrow$$

$$\varphi^*(h)^m = fe \text{ for some } e \in \Gamma(W)$$

$$\Downarrow$$

$$\underset{h^{m\cdot[L:K]}}{\overset{\|}{N_m(\varphi^*(h)^m)}} = N_m(fe) = \underbrace{N_m(f)}_{g} \underbrace{N_m(e)}_{\substack{\in K[X_1,\dots,X_n] \\ \text{as before}}}$$

$$\Downarrow$$

$$h^{m\cdot[L:K]} \in (g) \subseteq K[X_1, \dots, X_n]$$

$$\Downarrow$$

$$V(g) \subseteq V(h)$$

$$\Downarrow$$

$$h|_{V(g)} = 0 \quad \lightning$$

$\square$

Thm 2.83 Let W be an irred. alg. set and let V be an irreducible component of $V(f_1, \dots, f_r) \subseteq W$ for some $f_1, \dots, f_r \in \Gamma(W)$. Then, $\operatorname{codim}(V, W) \leq r$.

Pf Let V_1 be an irred. comp. of $V(f_1)$ containing V.

$\Rightarrow \dim(V_1) \geq \dim(W) - 1$.

Let V_2 be an irred. comp. of $V_1 \cap V(f_2)$ containing V.

$$\Rightarrow \dim(V_2) \geq \dim(V_1) - 1 \geq \dim(W) - 2.$$

$\vdots$

Let V_r — " — $V_{r-1} \cap V(f_r)$ containing V.

$\subseteq V(f_1, \dots, f_r)$

$\Rightarrow V = V_r$, $\dim(V_r) \leq \dim(W) - r$. $\square$

$\uparrow$

$V \subseteq V_r \subseteq V(f_1, \dots, f_r)$ irred. comp.

Rmk Even if $f_1, \dots, f_r \neq 0$ we might have $\operatorname{codim}(V, W) < r$ if $f_i|_{V_{i-1}} = 0$.

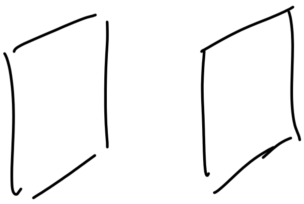
(E.g. if $f_1 = \dots = f_r$.)

Question (Matty) If $f_1, \dots, f_r$ are alg. indep. over K, is $\operatorname{codim}(V, W) = r$? No! Take $f_1 = x$, $f_2 = xy$.

Rmk $V(f_1, \dots, f_r)$ could be empty, even if

$r < \dim(W)$.

E.g. $\emptyset = V(X, X-1) \subseteq K^3$.



Rmk The Thm would fail for fields K that are not algebraically closed:

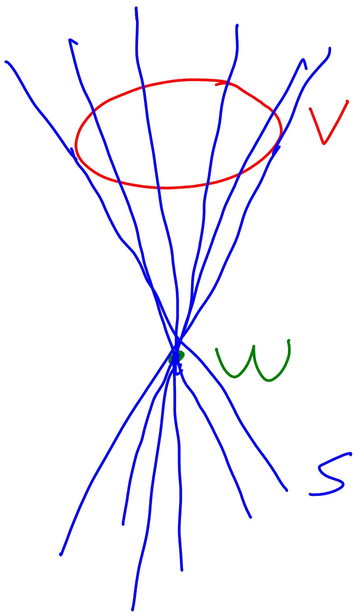
$\{(0,0)\} = V(x^2+y^2) \subseteq \mathbb{R}^2$.

2.20. Applications of dimensions, part 1

Thm 2.84 Let $V, W \subseteq K^n$ be irreducible of dimensions a, b and $S \subseteq K^n$ be the union of all straight lines $L \subseteq K^n$ joining a point $P \in V$ and a point $Q \in W$ with $P \neq Q$. (The set S is called the *join* of V, W.)

If $n \geq a+b+2$, then $S \subsetneq K^n$.

<u>Ese</u> $a=1, b=0, n=3$



<u>Pf</u> Consider the morphism

$$\varphi : V \times W \times K \longrightarrow K^n$$
$$(P, Q, t) \longmapsto \underbrace{tP + (1-t)Q}_{\text{parametrization of the line } PQ}$$

Its image contains S. (actually, the image is S, unless $V = W = \{P\}$, in which case $S = \emptyset$...)

$\Rightarrow$ By Lemma 2.64,

$$\dim(S) \leq \dim\left(\overline{\varphi(V \times W \times K)}\right) \leq \dim(V \times W \times K)$$
$$= \dim(V) + \dim(W) + \dim(K)$$
$$= a + b + 1 < n.$$

$\Rightarrow S \neq K^n$. $\square$

Thm 2.85 For any m points $P_1, \dots, P_m \in K^n$, if $m < \binom{d+n}{n}$, there is a polynomial $0 \neq f \in K[X_1, \dots, X_n]$ of degree $\leq d$ with $P_1, \dots, P_m \in V(f)$.

Ex $d = n = 2$, $m = 5$

$\Rightarrow \exists$ conic (or line) through any 5 points in K^2.

Pf Let F_d be the vector space of polynomials of degree $\leq d$.

Goal: $\exists\, 0 \neq f \in$ kernel of $F_d \subseteq \Gamma(K^n) \to \Gamma(\{P_1, \dots, P_m\})$
$$f \mapsto f|_{\{P_1, \dots, P_m\}}$$

$$\begin{aligned}\dim_K(F_d) &= \#\text{ monomials of degree} \leq d \\ &= \{(e_1, \dots, e_n) \mid e_1, \dots, e_n \geq 0,\ e_1 + \dots + e_n \leq d\} \\ &= \binom{d+n}{n}\end{aligned}$$

$$\underbrace{0 \dots 0}_{e_1} \mid \underbrace{0 \dots 0}_{e_2} \mid \ \dots \ \mid \underbrace{0 \dots 0}_{e_n} \mid \underbrace{0 \dots 0}_{d-(e_1+\dots+e_n)}$$

In total: d 0's, n 1's

$\dim_K(\Gamma(\{P_1, \dots, P_m\})) = m$. $\square$

Thm 2.86 For any points $P_1, \dots, P_m \in K^2$, there is an irreducible polynomial $0 \neq f \in K[X, Y]$ of degree $\leq m+2$ with $P_1, \dots, P_m \in V(f)$.

Pf The kernel T of $F_{m+1} \longrightarrow \Gamma(\{P_1, \dots, P_m\})$ has dimension $\dim(T) \geq \binom{m+4}{2} - m$.

Let $F_d' \subseteq F_d$ be the (algebraic!) set of pol. where at least one coeff. is 1.

Any reducible pol. $f \in F_{m+2}$ can be written as $f = gh$ with $g \in F_a$, $h \in F_b'$ where $a, b \geq 1$ with $a+b = m+2$.

The Zariski closure of the image of

$$\varphi_{a,b} : F_a \times F_b' \longrightarrow F_{m+2}$$
$$(g, h) \longmapsto gh$$

has dimension $\overline{\operatorname{im}(\varphi_{a,b})} \leq \dim(F_a \times F_b') = \binom{a+2}{2} + \binom{b+2}{2} - 1$

$$\Rightarrow \dim\Big(\overline{\bigcup_{\substack{a,b\geq 1:\\ a+b=m+2}} \operatorname{im}(\varphi_{a,b})}\Big)$$

$$= \max_{\substack{a,b\geq 1:\\ a+b=m+2}} \underbrace{\dim\big(\overline{\operatorname{im}(\varphi_{a,b})}\big)}_{\leq \binom{a+2}{2}+\binom{b+2}{2}-1}$$

$$= \dots = \frac{(m+5)(m+2)}{2} - ab + 1$$

$$\leq \frac{(m+5)(m+2)}{2} - (m+1) + 1$$

$$= \binom{m+3}{2} + 2$$

But $\binom{m+4}{2} - m - \binom{m+3}{2} - 2 = \binom{m+3}{1} - m - 2 = 1 > 0.$

$$\Rightarrow T \not\subseteq \bigcup_{a,b} \operatorname{im}(\varphi_{a,b})$$

$$\Rightarrow \exists\, f \in T \setminus \bigcup \operatorname{im}(\varphi_{a,b}) \qquad \square$$

Rmk There's room for improvement! If $f = gh$ with $P_1, \dots, P_m \in V(f)$, then $P_1, \dots, P_m \in V(g) \cup V(h)$, so we could fix a subset $S \subseteq \{P_1, \dots, P_m\}$ and consider only g, h with $S \subseteq V(g)$, $\{P_1, \dots, P_m\} \setminus S \subseteq V(h)$ (and take $\bigcup_{a,b,S} \dots$). ($\rightsquigarrow$ smaller dimension)