Cor 2.69 Let $\varphi: V \to W$ be a finite morphism. Then, any point $Q \in W$ has only finitely many preimages $P \in V$.

Pf Assume $Q \in \varphi(V)$.

$\Rightarrow$ The restriction $\varphi: \varphi^{-1}(Q) \longrightarrow \{Q\}$ is a surjective finite morphism.

$\Rightarrow \dim(\varphi^{-1}(Q)) = \dim(\{Q\}) = 0$

$\Rightarrow |\varphi^{-1}(Q)| < \infty$. $\square$

Thm 2.69 (Lying over property) Any dominant finite morphism $\varphi: V \to W$ is surjective.

Pf Let $Q \in W$ and let m be the maximal ideal of $\Gamma(W)$ corresponding to Q. (= the set of functions on W vanishing at Q)

Recall that $\varphi(P) = Q \Leftrightarrow \varphi(P) \in V(m) \Leftrightarrow P \in V(\underbrace{\varphi^*(m)}_{\subseteq \Gamma(V)})$

So $\varphi^{-1}(Q) = V(\varphi^*(m))$.

Let I be the ideal of $\Gamma(V)$ generated by $\varphi^*(m)$.

$\underset{\text{Nullstellensatz}}{\Rightarrow}$ If $\varphi(Q) = \phi$, then $I = \Gamma(V)$.

Let $\Gamma(V)$ be generated by $b_1, \dots, b_r$ as a $\varphi^*(\Gamma(W))$-module.

$\Rightarrow$ We can write any element of $\Gamma(V)$ as a lin. combination of $b_1, \dots, b_r$ with coeff. in $\varphi^*(\Gamma(W))$.

$\Rightarrow$ We can write any element of I as a lin. combination of $b_1, \dots, b_r$ with coeff. in $\varphi^*(m)$.

$\Rightarrow$ If $I = \Gamma(V)$, we can write

$$b_i = \varphi^*(p_{i1}) b_1 + \dots + \varphi^*(p_{ir}) b_r \text{ with}$$

$p_{i1}, \dots, p_{ir} \in \varphi^*(m)$.

$$\Rightarrow \underbrace{\begin{pmatrix} \varphi^*(p_{11}) & \cdots & \varphi^*(p_{1r}) \\ \vdots & & \vdots \\ \varphi^*(p_{r1}) & \cdots & \varphi^*(p_{rr}) \end{pmatrix}}_{M} \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix}$$

$$\Rightarrow (I_r - M) \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix} = 0$$

$r \times r$ identity matrise

$$\Rightarrow \det(I_r - M) = 0$$

as in the proof of Lemma 2.17

But all entries of M lie $\varphi^*(m)$.

Expanding the determinant, we see that

$0 = \det(I_r - M) = 1 + c$ for some $c \in \varphi^*(m)$.

$\Rightarrow 1 \in \varphi^*(m)$

$\Rightarrow 1 \in m$ $\lightning$

φ is dominant,
so φ^* is injective

$\square$

<u>Cor 2.70</u> Any finite morphism $\varphi: V \to W$ is <u>closed</u>: The image $\varphi(A) \subseteq W$ of every closed (=alg.) set $A \subseteq V$ is closed (=alg.)

<u>Ex</u> The proj. $K^2 \to K$ is <u>not</u> closed because the image of $\{(x,y) \mid xy = 1\}$ is not closed.

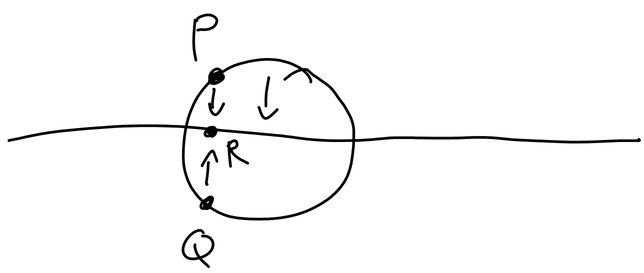
<u>Pf</u> $\varphi: A \longrightarrow \overline{\varphi(A)}$ is a dominant finite morphism, hence surjective.

$\Rightarrow \varphi(A) = \overline{\varphi(A)}$, so $\varphi(A)$ is closed. $\square$

Lemma 2.72 (Incomparability)

Let $\varphi: V \to W$ be a finite morphism and let $V_1 \subsetneq V_2 \subseteq V$ be alg. subsets with V_2 irreducible. Then $\varphi(V_1) \subsetneq \varphi(V_2) \subseteq W$.

Rmk This can fail if V_2 is reducible:



$V_1 = \{P\} \rightsquigarrow \varphi(V_1) = \{R\}$

$V_2 = \{P, Q\} \rightsquigarrow \varphi(V_2) = \{R\}$

Rmk We'll soon show that $V_1 \subsetneq V_2$, V_2 irreducible implies that

$$\begin{array}{ccc} \dim(V_1) & < & \dim(V_2) \\ \| & & \| \\ \dim(\varphi(V_1)) & & \dim(\varphi(V_2)) \end{array}$$

Pf W.l.o.g. $V = V_2$, $\varphi(V) = W$.

Let $0 \neq f \in \Gamma(V)$ with $f|_{V_1} = 0$.

Since $\Gamma(V)$ is an integral ext. of $\varphi^*(\Gamma(W))$, there is a monic polynomial equation

$$f^n + \varphi^*(c_{n-1}) f^{n-1} + \dots + \varphi^*(c_0) = 0$$

with $c_{n-1}, \dots, c_0 \in \Gamma(W)$. Pick one of smallest possible degree n.

$\Rightarrow \varphi^*(c_0)\big|_{V_1} = -f^n - \varphi^*(c_{n-1}) f^{n-1} - \dots - \varphi^*(c_1) f\big|_{V_1} = 0.$

If $\varphi(V_1) = W$, then $\Gamma(W) \xrightarrow{\varphi^*} \Gamma(V) \xrightarrow{i^*} \Gamma(V_1)$

$$g \mapsto g|_{V_1}$$

is injective because $V_1 \hookrightarrow V \longrightarrow W$ is dominant (actually surjective).

$\Longrightarrow c_0 = 0$

$\Longrightarrow f^{n-1} + \varphi^*(c_{n-1}) f^{n-2} + \dots + \varphi^*(c_1) = 0$

monic pol. eq. of degree $n-1 < n$. $\lightning$

$f \neq 0$ and $\Gamma(V)$ is an integral domain because V is irreducible

$\square$

<u>Lemma 2.73</u> Let $\varphi: V \to W$ be a dominant finite morphism and let B be an irreducible subset of W. Decompose $\varphi^{-1}(B)$ into irred. components: $\varphi^{-1}(B) = A_1 \cup \dots \cup A_r$.

Then, $\varphi(A_i) = B$ for <u>some</u> component A_i

<u>Pf</u> $B \underset{\substack{\uparrow \\ \varphi \text{ surjective}}}{=} \varphi(\varphi^{-1}(B)) = \underbrace{\varphi(A_1)}_{\text{closed}} \cup \dots \cup \underbrace{\varphi(A_r)}_{\text{closed}}$

B irred, $\varphi(A_1), \dots, \varphi(A_r)$ closed

$\Rightarrow \varphi(A_i) = B$ for some i.

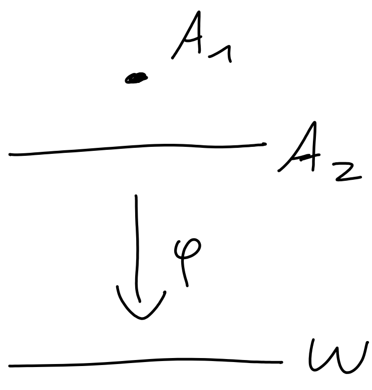
$\square$

Rmk: We might not have $\varphi(A_i) = B$ for <u>all</u> components A_i

Exe

$$V = \{(x,y) \mid y = 0\} \cup \{(0,1)\}$$

$$\begin{array}{ccc} \downarrow & (x,y) & \\ & \downarrow & \text{is finite} \\ B = W = K & x & \end{array}$$



Problem: V not irreducible

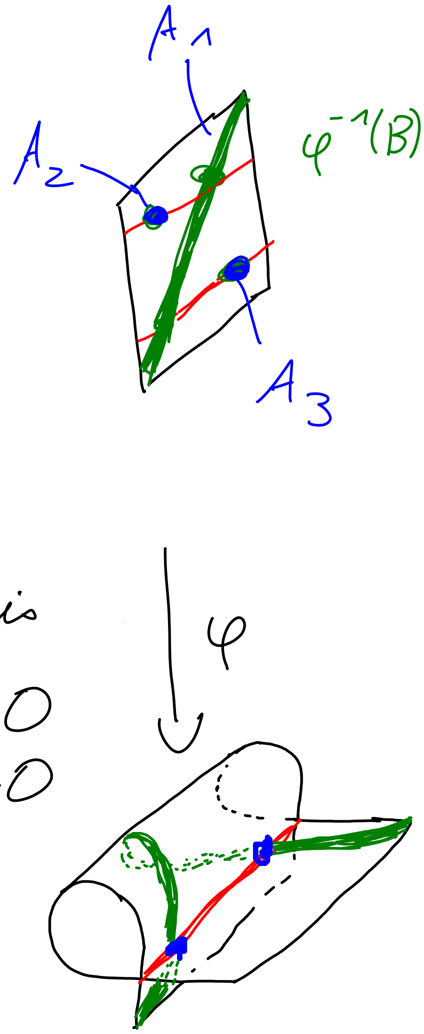
Exe

$$V = K^2 \qquad (t,u)$$

$$\varphi \downarrow \qquad\qquad \downarrow$$

$$(t^2-1,\ t(t^2-1),\ u)$$

$$W = \{(x,y,z) \mid x^2(x+1) = y^2\}$$



φ is finite because $T, U \in \Gamma(V)$ is integral over $\varphi^*(\Gamma(W))$:

$$T^2 - 1 - X = 0$$
$$U - Z = 0$$

$$B = \varphi(\underbrace{\{(t,u) \mid t = u\}}_{\cong K}) \text{ irreducible}$$

$$\varphi^{-1}(B) = \underbrace{\{(t,u) \mid t = u\}}_{A_1} \cup \underbrace{\{(1,-1)\}}_{A_2} \cup \underbrace{\{(-1,1)\}}_{A_3}$$

Problem: W not normal

Def An irreducible algebraic set $V \subseteq K^n$ is normal if the ring $\Gamma(V)$ is integrally closed in its field of fractions $K(V)$.

Ex K^n is normal.

Pf $\Gamma(V) = K[X_1, \ldots, X_n]$ is a unique factorization domain and hence integrally closed in its field of fractions by Thm 2.15. $\square$

Thm 2.74 (Going down)
Let V be an irreducible alg. set and let W be a normal alg. set. Let $\varphi : V \to W$ be a dominant finite morphism. Let B be an irreducible subset of W and decompose $\varphi^{-1}(B)$ into irred. comp.: $\varphi^{-1}(B) = A_1 \cup \ldots \cup A_r$.
Then $\varphi(A_i) = B$ for every component A_i.

Pf later...

2.16. Noether normalisation

Thm 2.75 (Noether normalisation)

Let R be a finitely generated ring ext. of K and an integral domain with field of fractions L.
($\Rightarrow L$ is a fin. gen. field ext. of K)
Let $n = \mathrm{trdeg}(L|K)$. Then, there are elements $a_1, \dots, a_n \in R$ such that R is an integral ext. of $K[a_1, \dots, a_n]$.

Rmk $a_1, \dots, a_n$ form a transcendence basis of L over K.

Pf of Rmk Every el. of R is algebraic over $K(a_1, \dots, a_n)$. $\Rightarrow$ Every el. of its field of fractions is alg. over $K(a_1, \dots, a_n)$. $\square$

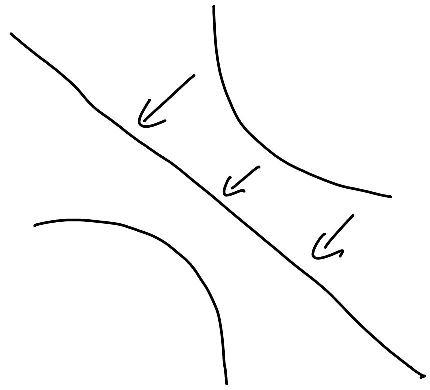
Cor 2.76 Let V be an irred. alg. set of dimension n. Then, there is a dominant finite morphism $V \longrightarrow K^n$.

Pf of Cor Apply Thm to $R = \Gamma(V)$ and use the finite morphism

$$\varphi : V \longrightarrow K^n \qquad \varphi^* : K[X_1, \dots, X_n] \hookrightarrow \Gamma(V)$$
$$p \longmapsto (a_1(p), \dots, a_n(p)) \qquad X_i \longmapsto a_i$$

$\square$

<u>Exe</u> The projections of $V = \{(x,y) \mid xy = 1\}$ onto the x- or y-axis is dominant, but not surjective!



However, the projection

$$\begin{aligned} V &\longrightarrow k \\ (x,y) &\longmapsto x+y \end{aligned}$$

is surjective:

The preimages of $t \in k$ are the points (x,y) where x is a solution to $x^2 - tx + 1 = 0$ and $y = t - x$.

It's actually a finite morphism.