Lemma 2.64 If there is a dominant rational map $\varphi: V \dashrightarrow W$, then $\dim(V) \geq \dim(W)$.

Pf Decompose V, W into irreducible components:

$$V = V_1 \cup \ldots \cup V_a, \quad W = W_1 \cup \ldots \cup W_b.$$

$$W = \overline{\varphi(V)} = \bigcup_{i=1}^{n} \overline{\varphi(V_i)}$$

$$\Rightarrow \underbrace{W_j}_{\text{irreducible}} = \bigcup_{i=1}^{n} \underbrace{(\overline{\varphi(V_i)} \cap W_j)}_{\text{closed}}$$

$\Rightarrow W_j = \overline{\varphi(V_{r_j})} \cap W_j$ for some $r_j \in \{1, \ldots, a\}$

$\Rightarrow W_j \subseteq \overline{\varphi(V_{r_j})}$

Since $\overline{\varphi(V_{r_j})} \subseteq W$ is irreducible, it is contained in some W_{s_j}.

$\Rightarrow W_j \subseteq \overline{\varphi(V_{r_j})} \subseteq W_{s_j}$

$\Rightarrow j = s_j, \quad W_j = \overline{\varphi(V_{r_j})}$

$\Rightarrow$ We obtain dominant morphisms

$$\varphi: V_{r_j} \longrightarrow W_j.$$

$\Rightarrow$ We can assume w.l.o.g. that V and W are irreducible.

$\varphi^*: K(W) \hookrightarrow K(V)$. If $a_1, \ldots, a_d$ is a transcendence basis of $K(W)$ (over K), then $\varphi^*(a_1), \ldots, \varphi^*(a_d)$ are still algebraically independent.

$\Rightarrow \dim(V) = \operatorname{trdeg}(K(V)|K) \geq d = \operatorname{trdeg}(K(W)|K) = \dim(W)$. $\square$

Lemma 2.65 If $V \subseteq W$, then $\dim(V) \leq \dim(W)$.

Pf W.l.o.g. V is irred. (replace by an irred. comp.)

W.l.o.g. W is irred. (replace by some irred. comp. containing V).

The inclusion $i : V \hookrightarrow W$ induces a surjective ring hom. $i^* : \Gamma(W) \to \Gamma(V)$.

$$f \mapsto f \circ i = f|_V$$

Since the elements of $\Gamma(V)$ generate $K(V)$ the field ext. of k, there are elements $a_1, \ldots, a_d \in \Gamma(V)$ which form a transcendence basis of $K(V)$.

Let $b_1, \ldots, b_d \in \Gamma(W)$ such that $i^*(b_i) = a_i$.

Then, $b_1, \ldots, b_d \in K(W)$ are still algebraically independent: If $f \in k[X_1, \ldots, X_d]$, $f(b_1, \ldots, b_d) = 0$, then

$$f(a_1, \ldots, a_d) = f(i^*(b_1), \ldots, i^*(b_d))$$

$$= i^*(f(b_1, \ldots, b_d)) = 0. \qquad \square$$

Thm 2.66 Let $V \subseteq K^n$ be an irreducible algebraic set. Then, $\dim(V)$ is the largest number $d \geq 0$ such that there is a dominant rational map $\varphi: V \dashrightarrow K^d$.

Pf Dominant rat. maps $\varphi: V \dashrightarrow K^d$ correspond to field homomorphisms

$$\varphi^*: \underset{\substack{\| \\ K(K^d)}}{K(X_1,\dots,X_d)} \hookrightarrow K(V)$$

which correspond to elements $a_i = \varphi^*(X_i)$ of $K(V)$. There is a well-defined field hom.

$$\cancel{\begin{array}{ccc} \mathbb{C}(X) & \longrightarrow & \mathbb{C} \\ X & \longmapsto & \pi \\ \frac{1}{X-\pi} & \longmapsto & \frac{1}{\pi-\pi} = \frac{1}{0} \end{array}}$$

$\varphi^*: K(X_1,\dots,X_d) \to K(V)$ sending X_i to $a_i \in K(V)$ if and only if the elements $a_i \in K(V)$ are algebraically independent over K (so that no nonzero denominator $0 \neq f \in K[X_1,\dots,X_d]$ can become $f(a_1,\dots,a_d) = 0$). $\square$

Rmk 2.67. There is in fact a dominant morphism $V \subseteq K^n \longrightarrow K^d$ for $d = \dim(V)$. Actually, there is a dominant projection $V \longrightarrow K^d$ onto a d-dimensional linear subspace H of K^n spanned by coordinate vectors!

Ex $n=2, d=1$: $\Rightarrow$ proj. onto x- or y-axis is dominant

$n=2, d=2$ $\Rightarrow$ The map $V \longrightarrow K^2$ is dominant

$$\Rightarrow \overline{V} = K^2 \underset{V \text{ closed}}{\Rightarrow} V = K^2$$

$n=3, d=2$ $\Rightarrow$ proj. onto xy- or xz- or yz-plane is dominant

Pf W.l.o.g. V is irred.
The field ext. $K(V)$ of K is generated by $X_1, \dots, X_n$. $\Rightarrow$ There is a transcendence basis of the form $X_{i_1}, \dots, X_{i_d}$. Then, the projection

$$\pi : V \longrightarrow K^d \qquad (x_1, \dots, x_n) \longmapsto (x_{i_1}, \dots, x_{i_d})$$

is dominant because

$$\pi^* : K[Y_1, \dots, Y_d] = \Gamma(K^d) \longrightarrow \Gamma(V), \quad Y_j \longmapsto X_{i_j}$$

is injective because $X_{i_1}, \dots, X_{i_d}$ are algebraically independent over K. $\square$

2.15. Finite morphisms

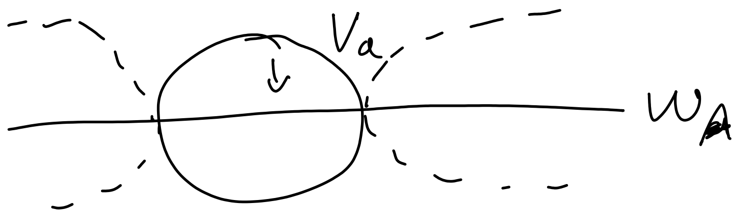
Some examples of dominant morphisms:

A) $\varphi_A : V_A = \{(x,y) \mid x^2 + y^2 = 1\} \longrightarrow W_A = K$

$\qquad (x,y) \longmapsto x$

with image $\varphi_A(V_A) = K = W_A$

The preimage $\varphi^{-1}(t) \subseteq V_A$ of any $t \in K$ consists of at most 2 (and at least 1) points: $(t, \pm\sqrt{1-t^2})$.



$\varphi_A^* : \Gamma(W_A) = K[T] \hookrightarrow K[X,Y]/(X^2+Y^2-1) = K[T][Y]/(T^2+Y^2-1)$

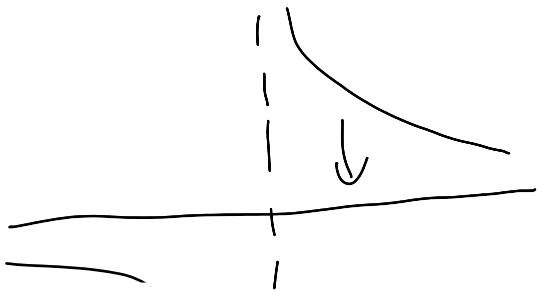
$\qquad T \longmapsto X \qquad\qquad \leftrightarrow T$

B) $\varphi_B : V_B = \{(x,y) \mid xy = 1\} \longrightarrow W_B = K$

$\qquad (x,y) \longmapsto x$

with image $\varphi_B(V_B) = K \setminus \{0\}$

The preimage $\varphi^{-1}(t) \subseteq V_B$ of any $t \in K$ consists of at most 1 point: $(t, \frac{1}{t})$.



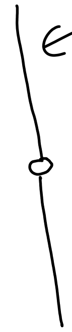
$\varphi_B^* : K[T] \hookrightarrow K[X,Y]/(XY-1) = K[T][Y]/(TY-1)$

$\qquad T \longmapsto X \qquad\qquad \leftrightarrow T$

c) $\varphi_C : V_C = K^2 \longrightarrow W_C = K^2$

$(x, y) \longmapsto (x, xy)$

with image $\varphi_C(V_C) = \{(t, u) \mid t \neq 0 \text{ or } u = 0\}$



← points not in the image

The preimage $\varphi^{-1}(t, u)$ consists of exactly 1 point $(t, \frac{u}{t})$ if $t \neq 0$ and infinitely many points $(0, *)$ if $t = u = 0$.

$$\varphi_C^* : K[T, U] \longrightarrow K[X, Y] \cong K[T, U][Y]/(TY - U)$$

$$\begin{array}{ccccc} T & \mapsto & X & \Longleftrightarrow & T \\ U & \mapsto & XY & \Longleftrightarrow & U \end{array}$$

What distinguishes A from B, C?

A) The ring ext. $\Gamma(V_A)$ of $\varphi_A^*(\Gamma(W_A))$ is integral since the polynomial $T^2 + Y^2 - 1$ is monic when considered a polynomial in Y with variables in $K[T]$. Hence, for any t, the polynomial $t^2 + Y^2 - 1 \in K[Y]$ still has degree 2 and therefore has 1 or 2 roots $y \in K$.

B) The pol. $TY - 1$ is not monic "in Y" and for some $t (= 0)$, the pol. $tY - 1$ is -1, which has no root in y.

C) The pol. $TY - U$ is not monic "in Y" and for $t = u = 0$, the $tY - u$ is 0, which has ∞ many roots $y \in K$.

Def A morphism $\varphi: V \longrightarrow W$ is <u>finite</u> if the ring extension $\Gamma(V)$ of $\varphi^*(\Gamma(W))$ is module-finite (or, equivalently, integral).

Ex φ_A is finite.

Ex Let $V \subseteq W$. Then, the inclusion morphism $i: V \hookrightarrow W$ is finite (because $i^*: \Gamma(W) \longrightarrow \Gamma(V)$ is surjective).

Rmk The composition of two finite morphisms is finite.

Pf This follows from the transitivity of module-finiteness (or integrality). $\square$

Rmk In particular, the restriction of a finite morphism $V \longrightarrow W$ to an alg. subset $V' \subseteq V$ is finite. (It's the composition $V' \hookrightarrow V \longrightarrow W$.)

Rmk $\varphi: V \longrightarrow W \subseteq K^m$ is finite if and only if $\varphi: V \longrightarrow K^m$ is finite.

Pf $\varphi^*(\underbrace{\Gamma(W)}_{K[X_1,\dots,X_m]/I(W)}) = \varphi^*(\underbrace{\Gamma(K^m)}_{K[X_1,\dots,X_m]})$. $\square$

Thm 2.68 If $\varphi : V \to W$ is a dominant finite morphism, then $\dim(V) = \dim(W)$.

Pf Decompose into irred. comp.:

$$V = V_1 \cup \dots \cup V_a, \quad W = W_1 \cup \dots \cup W_b.$$

As in the proof of Lemma 2.64, we obtain dom. finite morphisms $\varphi : V_{r_j} \longrightarrow W_j$ for $j = 1, \dots, b$.

$\Rightarrow$ We can assume w.l.o.g. that V, W are irreducible, so we get a field hom. $\varphi^* : K(W) \hookrightarrow K(V)$.

φ finite $\Longrightarrow$ $\Gamma(V)$ integral ext. of $\varphi^*(\Gamma(W))$.

$\Rightarrow$ $K(V)$ algebraic over $\varphi^*(K(W))$.

If $b_1, \dots, b_d$ is a transcendence basis of $K(W)$, then $\varphi^*(b_1), \dots, \varphi^*(b_d)$ is a transcendence basis of $K(V)$.

$\square$