# 2.14. Dimension and transcendence degree

**Def** Let $L|K$ be a field extension. Elements $a_1, \dots, a_n \in L$ <u>algebraically dependent</u> over $K$ if there is a polynomial $0 \neq f \in K[X_1, \dots, X_n]$ such that $f(a_1, \dots, a_n) = 0$.

($\Leftrightarrow$ $a_1$ algebraic if $n=1$)

**Ex** $X_1, \dots, X_n \in K(X_1, \dots, X_n)$ are algebraically independent over $K$.

**Ex** $X, Y \in K(V(X^2 - Y^3))$ are algebraically dependent over $K$: $X^2 - Y^3 = 0$ in $K(V(X^2-Y^3))$.

**Rmk** $\pi, e \in \mathbb{C}$ are transcendental over $\mathbb{Q}$.
It is unknown whether $\pi, e$ are algebraically independent over $\mathbb{Q}$.

**Thm 2.57** $a_1, \dots, a_n \in L$ are algebraically dependent if and only if some $a_i$ is algebraic over $K(a_1, \dots, a_{i-1})$.

**Analogy** Let $V$ be a $K$-vector space. Then, $v_1, \dots, v_n \in V$ are linearly dependent if and only if some $v_i$ is contained in the span of $v_1, \dots, v_{i-1}$.

Pf "$\Leftarrow$" $a_i$ algebraic over $K(a_1,\dots,a_{i-1})$.

$\Rightarrow f(a_i)=0$ for some $0\neq f\in K(a_1,\dots,a_{i-1})[T]$

Clear out denominators to make

$0\neq f\in K[a_1,\dots,a_{i-1}][T]$.

"$\Rightarrow$" Let $0\neq f\in K[X_1,\dots,X_n]$, $f(a_1,\dots,a_n)=0$.

If $g(X_n)=f(a_1,\dots,a_{n-1},X_n)\in K(a_1,\dots,a_{n-1})[T]$ is not the zero polynomial, then $g(a_n)=0$ is a pol. eq. satisfied by $a_n$ with coeff. in $K(a_1,\dots,a_{n-1})$, so $a_n$ is algebraic over $K(a_1,\dots,a_{n-1})$.

$\rightsquigarrow$ Assume $g(X_n)$ is the zero polynomial.

Let $f(X_1,\dots,X_n)=\sum_i f_i(X_1,\dots,X_{n-1})\cdot X_n^i$

with $f_j(X_1,\dots,X_{n-1})$ not the zero polynomial for some $i$

But because $g(X_n)$ is the zero polynomial, we have

$f_j(a_1,\dots,a_{n-1})=0$.

$\Rightarrow a_1,\dots,a_{n-1}$ are algebraically dependent.

Proceed by induction over $n$. $\square$

Def Elements $a_1, \dots, a_n \in L$ form a transcendence basis of $L$ over $K$ if they are algebraically independent and $L$ is an algebraic ext. of $K(a_1, \dots, a_n)$.

Rmk A transcendence basis is a maximal list of algebraically independent elements.

Ex $X_1, \dots, X_n$ form a transcendence basis of $K(X_1, \dots, X_n)$.

Lemma 2.58 ("Exchange lemma")

If $a_1, \dots, a_n \in L$ are alg. independent and $L$ is algebraic over $K(b_1, \dots, b_m)$ (with $b_1, \dots, b_m$), then there are indices

$1 \leq i_1 < \dots < i_r \leq m$ $(r \geq 0)$ such that

$a_1, \dots, a_n, b_{i_1}, \dots, b_{i_r}$ form a transcendence basis of $L$ over $K$.

Pf Choose any maximal alg. independent sublist of $a_1, \dots, a_n, b_1, \dots, b_m$ among those containing $a_1, \dots, a_n$. The remaining $b_j$ have to be algebraic over

$K$( the list you got).

$\Rightarrow K$( the list) $= L$. $\square$

Cor 2.59 Any finitely generated field extension has a transcendence basis.

Thm 2.60 If $a_1, \dots, a_n$ is a transcendence basis of $L$ over $K$ and $b_1, \dots, b_m \in L$ are algebraically independent, then $n \geq m$.

Cor 2.61 Any two transcendence bases of $L$ over $K$ have the same size, called the transcendence degree $\operatorname{trdeg}(L|K)$ of $L$ over $K$.

Rmk $\operatorname{trdeg}(L|K) = 0 \iff L$ is an algebraic extension of $K$

Pf of Thm 2.60

Use induction over $n$.

$n=0$: $\Rightarrow L$ is alg. over $K$ $\Rightarrow$ there are no algebraically independent elements. $\Rightarrow m=0$

$n-1 \to n$: Let (w.l.o.g.) $b_1, a_1, \dots, a_r$ be a transcendence basis from the exchange lemma.

$\Rightarrow r \neq n$ because $b_1, a_1, \dots, a_n$ aren't alg. indep. because $a_1, \dots, a_n$ is a transcendence basis

$a_1, \dots, a_r \in L$ form a transcendence basis of $L$ over $K(b_1)$.

$b_2, \dots, b_m \in L$ are alg. independent over $K(b_1)$.

Apply the induction hypothesis to the extension $L$ of $K(b_1)$.

$\Rightarrow n-1 \geq r \geq m-1 \quad \Rightarrow n \geq m$.

$\square$

Ex $X$ is a transcendence basis of $K(V(X^2-Y^3))$

$\Uparrow$ $Y$ —"— —

- No nonzero pol. $f(X) \in K[X]$ becomes zero in $K(V(X^2-Y^3))$ because it doesn't become zero in $\Gamma(V(X^2-Y^3)) = K[X,Y]/(X^2-Y^3)$ because $f(X) \notin (X^2-Y^3)$.

  $\Rightarrow X$ is alg. independent

- $X, Y$ are algebraically dependent
- $X, Y$ generate the field extension $K(V(X^2-Y^3))$

$\Rightarrow \operatorname{trdeg}(K(V(X^2-Y^3))) = 1$.

Thm 2.61 If $L$ is a finitely generated field extension of $K$ and $M$ is a finitely generated field extension of $L$, then

$$\begin{array}{c} M \\ | \\ L \\ | \\ K \end{array} \qquad \operatorname{trdeg}(M|K) = \operatorname{trdeg}(M|L) + \operatorname{trdeg}(L|K)$$

Pf If $a_1, \dots, a_n$ is a transcendence basis of $M/L$
and $b_1, \dots, b_m$ " — $L|K$,
then $a_1, \dots, a_n, b_1, \dots, b_m$ " — $M|K$.

"$\square$"

Def The dimension $\dim(V)$ of an irreducible algebraic set $V \subseteq K^n$ is the transcendence degree of $K(V)$ over $K$.

Ex $\dim(\underbrace{\mathbb{A}^n_K}_{K^n}) = n$

Cor $\mathbb{A}^n = K^n$ is not isomorphic (or even birational) to $\mathbb{A}^m = K^m$ for any $n \neq m$.

## Analogy from topology

There is no homeomorphism between $\mathbb{R}^n$ and $\mathbb{R}^m$ for $n \neq m$. In fact, there is no homeomorphism between an open subset $\emptyset \neq U \subseteq \mathbb{R}^n$ and an open subset $\emptyset \neq V \subseteq \mathbb{R}^m$!

**Thm 2.62** The dimension of an irreducible algebraic subset $V = V(f) \subsetneqq K^n$ defined by a single (irreducible) polynomial

$$0 \neq f \in K[X_1, \dots, X_n] \text{ is } \dim(V) = n-1.$$

**Pf** W.l.o.g. the variable $X_n$ occurs in $f$.

$$\Rightarrow X_1, \dots, X_{n-1} \in \underset{\displaystyle \overset{\cap}{K(V)}}{\Gamma(V)} = K[X_1, \dots, X_n]/(f)$$

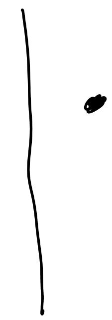
form a transcendence basis of $K(V)$ over $K$.

(They are algebraically independent because $(f)$ contains no polynomial in just the variables $X_1, \dots, X_{n-1}$.
But $X_1, \dots, X_n$ are not algebraically independent.) $\square$

Remark Why only define $\dim(V)$ when $V$ is irreducible?

A) If $V$ isn't irreducible, $\Gamma(V)$ is not an integral domain, and there is no field of fractions $K(V)$!

B) Say $V = \{(x,y) \in K^2 \mid x = 0\} \cup \{(1,2)\}$.



We define the dimension of a reducible alg. set $V \subseteq K^n$ to be
$$\dim(V) = \max(\dim(W_1), \ldots, \dim(W_m)),$$
where $W_1, \ldots, W_m$ are the irreducible components of $V$.

And $\dim(\emptyset) = -\infty$.

Thm 2.63 An alg. subset $\emptyset \neq V \subseteq K^n$ has dimension 0 if and only if $|V| < \infty$.

Pf W.l.o.g. V is irreducible.

"$\Leftarrow$" If $|V| < \infty$, then $|V| = 1$, $V = \{p\}$.

$\Rightarrow \Gamma(V) = K$, $K(V) = K$.

trdeg $= 0$

"$\Rightarrow$" $\dim(V) = 0 \Rightarrow K(V)$ is an alg. ext. of $K$

$\Rightarrow K(V) = K \Rightarrow \Gamma(V) = K$

K algebraically closed

$\Gamma(V) = K[X_1, \dots, X_n]/I(V)$

$\Rightarrow |V| = 1$. $\square$